Descriptive Set Theory and Dimension

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All spaces are separable and metrizable.

Standard Notation for Borel Classes:

Π^0_α is the multiplicative class,

Σ^0_α is the additive class

of absolute Borel spaces corresponding to a countable ordinal α.

The classes Π^0_α, Σ^0_α are dual each to the other.

In particular:

Π^0_1 is the class of compact spaces,

Σ^0_2 is the class of σ-compact spaces,

Π^0_2 is the class of Polish spaces,

Π^0_3 is the class of absolute $F_{\sigma\delta}$-sets.

Motivation:

Theorem (Wadge, Louveau, Saint Raymond)

Let $\Gamma \in \{\Pi^0_\xi, \Sigma^0_\xi : \xi \geq 3\}$ be a Borel class and $\check{\Gamma}$ be its dual.
For a space $\tilde{X} \in \check{\Gamma}$ and a Borel subset $X \subset \tilde{X}$ TFAE:

0) $X \notin \check{\Gamma}$;   0') $\tilde{X} \setminus X \notin \Gamma$;

1) For any 0-dimensional space $C \in \Gamma$ there is a closed embedding $f : C \hookrightarrow X$;

2) For any subspace $C \in \Gamma$ of a 0-dimensional compact space K there is an embedding $e : K \hookrightarrow \tilde{X}$ with $e^{-1}(X) = C$;

3) For a subspace $C \in \Gamma$ of a 0-dimensional compact space K there is a map $e : K \hookrightarrow \tilde{X}$ with $e^{-1}(X) = C$;

4) For a 0-dimensional space $C \in \Gamma$ there is a perfect map $f : C \to X$.

A map $f : X \to Y$ is *perfect* if the preimage $f^{-1}(K)$ of any compact $K \subset Y$ is compact.

Problem: Do (1)–(4) remain equivalent if one replaces "0-dimensional" with "n-dimensional"?

Some Terminology

Definition 1: Let Γ be a class of spaces. A space X is called

• Γ-*universal* if for each space $C \in \Gamma$ there is a closed embedding $f : C \hookrightarrow X$;

• Γ-*preuniversal* if each space $C \in \Gamma$ there is a perfect map $f : C \hookrightarrow X$.

Similar notions can be introduced for pairs of spaces $(\tilde{X}, X)$ where $X \subset \tilde{X}$.

Definition 2: Let $\vec{\Gamma}$ be a class of pairs. A pair $(\tilde{X}, X)$ is called

• $\vec{\Gamma}$-*universal* if for each pair $(K, C) \in \vec{\Gamma}$ there is a closed embedding $f : K \hookrightarrow \tilde{X}$ such that $f^{-1}(X) = C$;

• $\vec{\Gamma}$-*preuniversal* if each pair $(K, C) \in \vec{\Gamma}$ there is a perfect map $f : K \to \tilde{X}$ with $f^{-1}(X) = C$.

For two classes of spaces $\mathcal{K}, \mathcal{C}$ let

$$(\mathcal{K}, \mathcal{C}) = \{(K, C) : K \in \mathcal{K},\ C \in \mathcal{C}\}.$$

For a class Γ of spaces, let

$$\Gamma[n] = \{C \in \Gamma : \dim C \leq n\}.$$

In particular, $\Pi^0_1[n]$ will stand for the class of metrizable compact spaces of dimension $\leq n$.

Theorem (Wadge, Louveau, Saint Raymond)
(expanded version)

Let $\Gamma \in \{\Pi^0_\xi, \Sigma^0_\xi : \xi \geq 3\}$ be a Borel class and $\check{\Gamma}$ be its dual Borel class. For a space $\tilde{X} \in \check{\Gamma}$ and a Borel subspace $X \in \tilde{X}$ TFAE:

0) $X \notin \check{\Gamma}$;

0') $\tilde{X} \setminus X \notin \Gamma$;

1) X is $\Gamma[0]$-universal;

1') $\tilde{X} \setminus X$ is $\check{\Gamma}[0]$-universal;

2) X is $\Gamma[0]$-preuniversal;

2') $\tilde{X} \setminus X$ is $\check{\Gamma}[0]$-preuniversal;

3) $(\tilde{X}, X)$ is $(\Pi^0_1[0], \Gamma)$-universal;

3) $(\tilde{X}, \tilde{X} \setminus X)$ is $(\Pi^0_1[0], \check{\Gamma})$-universal;

4) $(\tilde{X}, X)$ is $(\Pi^0_1[0], \Gamma)$-preuniversal.

4') $(\tilde{X}, \tilde{X} \setminus X)$ is $(\Pi^0_1[0], \check{\Gamma})$-preuniversal.

Problem: What about higher-dimensional version of this theorem?

Theorem (Banakh, 1996):

Let $\Gamma \in \{\Pi^0_\xi, \Sigma^0_\xi : \xi \geq 3\}$ be a Borel class, $\check{\Gamma}$ be its dual Borel class, and $n \in \omega \cup \{\infty\}$. For a Polish space $\tilde{X}$ and a subspace $X \in \tilde{X}$ TFAE:

1) X is $\Gamma[n]$-universal;

1') $\tilde{X} \setminus X$ is $\check{\Gamma}[n]$-universal;

2) X is $\Gamma[n]$-preuniversal;

3') $\tilde{X} \setminus X$ is $\check{\Gamma}[n]$-preuniversal;

4) $(\tilde{X}, X)$ is $(\Pi^0_1[n], \Gamma)$-universal;

4') $(\tilde{X}, \tilde{X} \setminus X)$ is $(\Pi^0_1[n], \check{\Gamma})$-universal;

5) $(\tilde{X}, X)$ is $(\Pi^0_1[n], \Gamma)$-preuniversal.

5') $(\tilde{X}, \tilde{X} \setminus X)$ is $(\Pi^0_1[n], \check{\Gamma})$-preuniversal.

Open Problem: Is this theorem true if $\tilde{X} \in \check{\Gamma}$? (Yes, if $n = 0$ and X is Borel).

This theorem is a partial case of a more general result treating the universality for sufficiently rich classes of pairs.

Definition: A class $\vec{\Gamma}$ of pairs is:

• *compact* if for each pair $(K, C) \in \Gamma$ the space K is compact;

• 2^ω-*stable* if $(K, C) \in \vec{\Gamma} \Rightarrow (2^\omega \times K, 2^\omega \times C) \in \vec{\Gamma}$;

• F_σ-*additive* if for each $(K, C) \in \vec{\Gamma}$ and each F_σ-subset $A \subset K$ we get $(K, C \cup A) \in \vec{\Gamma}$;

• G_δ-*multiplicative* if for each $(K, C) \in \vec{\Gamma}$ and each G_δ-subset $G \subset K$ we get $(K, G \cap A) \in \vec{\Gamma}$.

Remark: For any Borel class $\Gamma \in \{\Pi^0_\xi, \Sigma^0_\xi : \xi \geq 3\}$ and any $n \in \{\omega \cup \{\infty\}$ the class $(\Pi^0_1[n], \Gamma)$ is compact, 2^ω-stable, F_σ-additive, and G_δ-multiplicative.

Theorem (Banakh, 1996):

Given a compact 2^ω-stable F_σ-additive G_δ-multiplicative class of pairs $\vec{\Gamma}$ let

$\Gamma = \{C : (K, C) \in \vec{\Gamma}\}$ and
$\check{\Gamma} = \{K \setminus C : (K, C) \in \vec{\Gamma}\}$.

For a Polish space $\tilde{X}$ and a subspace $X \in \tilde{X}$ TFAE:

1) X is Γ-universal;
1') $\tilde{X} \setminus X$ is $\check{\Gamma}$-universal;

2) X is Γ-preuniversal;
3') $\tilde{X} \setminus X$ is $\check{\Gamma}$-preuniversal;

4) $(\tilde{X}, X)$ is $\vec{\Gamma}$-universal;
5) $(\tilde{X}, X)$ is $\vec{\Gamma}$-preuniversal.

Some corollaries

For classes $\vec{\Gamma}$, Γ, and $\check{\Gamma}$ from the preceding theorem and a Γ-universal space $X \in \Gamma$ any perfect continuous image Y of X contains a closed topological copy of X and hence $\dim Y \geq \dim X$.

So, the dimension of Γ-universal spaces cannot be lowered by perfect maps.

In particular, any perfect image of the n-dimensional Nöbeling space (which is $\Pi^0_2[n]$-universal) has dimension $\geq n$.

Constructions increasing the Borel complexity

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1. Spaces of measures

Theorem (??, Banakh-Radul, 1997)
Let $P_R(X)$ be the space of probability Radon measures on X, endowed with the weak-star topology.
If X is $\Sigma^0_{<\xi}[0]$-universal, then
$P_R(X)$ is $\Pi^1_\xi[\infty]$-universal.

Theorem (??, Banakh-Cauty, 1997)
Let $P_\beta(X)$ be the space of probability measures with compact support on X.
If X is $\Sigma^1_n[0]$-universal, then
$P_\beta(X)$ is $\Pi^1_{n+1}[\infty]$-universal.

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2. Hyperspaces

Theorem (??, Banakh, 1997)
Let X be a connected locally path-connected space and 2^X be the hyperspace of non-empty compact subsets of X, endowed with the Vietoris topology.
If X is $\Sigma^1_n[1]$-universal, then
2^X is $\Pi^1_{n+1}[\infty]$-universal.

3. Countable powers

Trivial Observation:

1) The interval $\mathbb{I} = [0,1]$ is a 1-dimensional compact space with $\Pi^0_1[\infty]$-universal power $\mathbb{I}^\omega$.

2) The real line $\mathbb{R}$ is a Polish 1-dimensional having $\Pi^0_1[\infty]$-universal countable power $\mathbb{R}^\omega$.

What about higher Borel classes?

Theorem (Banakh-Cauty, 2001):

1) For any finite-dimensional space X the countable power X^ω is not $\Sigma^0_2[s.c.d]$-universal, where $\Sigma^0_2[s.c.d]$ is the class of spaces that are countable unions of finite-dimensional compacta;

2) For any countable-dimensional space X the power X^ω is not Σ^0_3-universal.

4. Finite powers

Theorem (Menger-Nöbeling-Pontryagin-Lefschetz)
If X is contains an arc, then
X^{2n+1} is $\Pi^0_1[n]$-universal.

Theorem (Banakh-Cauty-Truščak-Zdomsky, 2006)
If X is a locally path-connected nowhere locally compact space, then X^{n+1} is $\Pi^0_2[n]$-universal.

Theorem (Banakh-Cauty, 2005)
If X is a meager locally path-connected space, then X^{2n+1} is $\Sigma^0_2[n]$-universal.

Theorem (Bowers, 1985; Banakh-Cauty, 2005)
For a Borel class $\Gamma \in \{\Pi^0_1, \Pi^0_2, \Sigma^0_2\}$ there is a 1-dimensional space $X \in \Gamma$ such that each power X^{n+1} is $\Gamma[n]$-universal.

Such a space X is a subspace of a dendrite with dense set of end-points.

Open Problem. Which Borel classes Γ do contain a 1-dimensional space X with $\Gamma[n]$-universal powers X^{n+1}?

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Thank You!