Covering reals by translations of a compact set

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Problem (Gruenhage): Given a compact subset of the real line K is it consistent that the real line is covered by $< 2^{\aleph_0}$ translations of K? Or more generally, if K is a compact subset of a Polish group.

Obstruction: there exists a perfect set P such that for every x, $(K + x) \cap P$ is countable.

This obstruction is a $\mathbf{\Sigma}^1_2$ property of K.
To see this note that K is small iff $\exists P$

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- If C is the ordinary Cantor set then $\mathbf{R}$ is not covered by $< 2^{\aleph_0}$ translations of C. (Gruenhage)
- if C has packing dimension < 1 then $\mathbf{R}$ is not covered by $< 2^{\aleph_0}$ translations of C. (Darji-Keleti)

- if K is not meager then $\mathbf{R}$ is covered by countably many translations of K (folklore),
- if K has positive measure then $\mathbf{R}$ is covered by non($\mathcal{N}$) translations of K (folklore),
- there is a compact set K of measure zero such that $\mathbf{R}$ is covered by cof($\mathcal{N}$) translations of K (Elekes-Steprans),
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Digression

- Let $\mathbf{G}$ be a Polish group and let $\mathrm{cov}^{\star}_{\mathbf{G}}(\mathcal{M})$ be the minimal cardinality of set $X \subseteq \mathbf{G}$ such that for some closed nowhere dense set M, $X + M = \mathbf{G}$. The value of $\mathrm{cov}^{\star}_{\mathbf{G}}(\mathcal{M})$ depends on $\mathbf{G}$. (Miller-Steprans)
- Suppose that given an uncountable set $X \subseteq \mathbf{R}$ we can find a compact measure zero set K such that $\bigcup_{x \in X}(K + x) = \mathbf{R}$. then Borel Conjecture + Dual Borel Conjecture holds.

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Definition

A perfect set $K \subseteq 2^\omega$ is big if for every $n \in \omega$ there exists $j_n \in \omega$ such that for $X \subseteq 2^\omega$ and $x \in 2^\omega$, if

1. $|X| \leq n$,
2. $(2^\omega \setminus K) + X \neq 2^\omega$,
3. $x \restriction j_n \in X \restriction j_n$,

then

$$(2^\omega \setminus K) + (X \cup \{x\}) \neq 2^\omega.$$

We say that K is big* if $K \cap [s]$ is big for every $s \in 2^{<\omega}$ such that $K \cap [s] \neq \emptyset$.

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Theorem

If K is big*, then there is a ccc-extension of the universe in which 2^ω is covered by $< 2^{\aleph_0}$ translations of K.

Let $\mathbb{Q} = \{q \in 2^\omega : \forall^\infty n \; q(n) = 0\}$.

Lemma

Suppose that $K \subseteq 2^\omega$ is big*. There exists a ccc forcing notion $\mathbb{P}_K$ which adds real $z_K \in 2^\omega$ such that

$$\Vdash_{\mathbb{P}_K} \forall x \in 2^\omega \cap \mathbf{V} \; \exists q \in \mathbb{Q} \; x \in K + z_K + q.$$

Let $\mathbb{P}_K$ be the collection of pairs (t, X) such that

1. $t \in 2^{<\omega}$ and X is a finite subset of 2^ω,
2. $((2^\omega \setminus K) + X) \cap [t] \neq \emptyset$.

For $(t_0, X_0), (t_1, X_1) \in \mathbb{P}_K$, we put $(t_1, X_1) \geq (t_0, X_0)$ if $t_0 \subseteq t_1$ and $X_0 \subseteq X_1$.

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Suppose that $K \subseteq 2^\omega$. If K is not small, then it is consistent that 2^ω is covered by $< 2^{\aleph_0}$ translations of K.

The same holds for subsets K of locally compact abelian Polish groups.

If K is not small then in the Sacks model $\mathbf{V}^{\mathbb{S}_{\omega_2}}$,

$$\forall x \in 2^\omega\ \exists z \in \mathbf{V} \cap 2^\omega\ x \in K + z.$$

The following is a technical restatement of this fact.

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Suppose that $p \Vdash_{\mathbb{S}_{\omega_2}} \dot{x} \in 2^\omega \setminus \mathbf{V}$. Then there exists $p' \geq p$ and a perfect set $P \subseteq 2^\omega$ such that for every perfect set $Q \subseteq P$ there exists $q \geq p'$ such that $q \Vdash \dot{x} \in Q$.

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Suppose that K is not small and let $p \Vdash_{\mathbb{S}_{\omega_2}} \dot{x} \in 2^\omega$. Find $p' \geq p$ and P. Since K is not small there is $z \in 2^\omega$ such that $P \cap (K + z)$ is uncountable. Let $Q \subseteq P \cap (K + z)$ be a perfect set. It follows that there is $q \geq p'$ such that $q \Vdash_{\mathbb{S}_{\omega_2}} \dot{x} \in Q \subseteq K + z$.

CPA $\iff 2^{\aleph_0} = \aleph_2$ and for every "appropriately" dense family $\mathcal{E}_0 \subset \mathbb{S}$ there is an $\mathcal{E}_0 \subset \mathcal{E}$ such that $|\mathcal{E}_0| \leq \aleph_1$ and $|\mathbf{R} \setminus \bigcup \mathcal{E}_0| \leq \aleph_1$.

Theorem

Assume $\mathbf{CPA}_{prism}$. *Then if* K *is not small then* $\mathbf{R}$ *is covered by* $\aleph_1$ *translations of* K.

Examples of sets which are not small

Let $\{I_n : n \in \omega\}$ be a partition of ω into finite sets of increasing size and let $K_n \subset 2^{I_n}$. Consider sets of form $K = \prod_n K_n$.

Lemma

If $\lim_n \frac{|K_n|}{|2^{I_n}|} = 1$ *then* K *is big**.

Lemma (Elekes-Toth)

Suppose that $I \subseteq \omega$ *is finite,* $n \in \omega$ *and* $C \subset 2^I$ *is such that* $\frac{|C|}{2^{|I|}} \geq 1 - \frac{1}{n+1}$. *For any* $X \subseteq 2^I$ *of size* $\leq n$ *there exists* $t \in 2^I$ *such that* $t + X \subseteq C$.

Choose K_n's such that

$$1 - \frac{1}{n+1} \leq \frac{|K_n|}{|2^{I_n}|} \leq 1 - \frac{1}{2n+1}.$$

Then $K = \prod_n K_n$ has measure zero.

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If $\lim_n \frac{|K_n|}{|2^{I_n}|} < 1$ then K may be small or big*, depending on the choice of K_n's.

Lemma

Fix $\varepsilon > 0$. There exists $K_n \subseteq 2^{I_n}$ such that for each n, $|K_n|/2^{|I_n|} \leq \varepsilon$ and $K = \prod_n K_n$ is small.

Lemma

Fix a sequence of positive reals $\{\varepsilon_n : n \in \omega\}$. There exists a sequence $K_n \subseteq 2^{I_n}$ such that for each n, $|K_n|/2^{|I_n|} \leq \varepsilon_n$ and $K = \prod_n K_n$ is big*.

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Theorem (Bartoszynski-Shelah)

Suppose that $m \in \omega$ and $0 < \delta < \varepsilon < 1$ are given. There exists $n \in \omega$ such that for every finite set $I \subseteq \omega$ of size at least n, there exists a set $C \subseteq 2^I$ such that $\varepsilon + \delta \geq |C| \cdot 2^{-|I|} \geq \varepsilon - \delta$ and for every set $X \subseteq 2^I$, $|X| \leq m$

$$\left| \frac{|\bigcap_{s \in X}(C + s)|}{2^{|I|}} - \varepsilon^{|X|} \right| < \delta.$$

Note that the theorem says that we can choose C is such a way that for any sequences $s_1, \ldots, s_m \in 2^I$ the sets $s_1 + C, \ldots, s_m + C$ are probabilistically independent with error δ.

Thus, if we choose δ to be much smaller than ε^m, then if $|X| < m$ it follows that $\bigcap_{s \in X}(C + s) \neq \emptyset$. In particular, if $t \in \bigcap_{s \in X}(C + s)$ then $X \subseteq C + t$.

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