The descriptive set theory of orbit equivalence

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Borel complexity

Suppose that $\mathcal{A}$ is a standard Borel space and E is an equivalence relation on $\mathcal{A}$.

Definition

E on $\mathcal{A}$ is Borel reducible to F on $\mathcal{B}$, denoted $E \leq_B F$ if there is a Borel map $\phi\colon \mathcal{A} \to \mathcal{B}$ such that

$$xEy \iff \phi(x)F\phi(y).$$

This is meant to reflect that F is "more complex" than E and that the points of $\mathcal{A}$ can be classified up to E-equivalence by a Borel assignment of invariants that are F-equivalence classes.

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E is smooth if $E \leq_B id(\mathbb{R})$.

Example: If we consider E to be the equivalence relation of similarity on the space of $n \times n$ matrices, then we can let $f(A)$ be the Jordan form of A.

E_0

E_0 is the equivalence relation given by eventual agreement on $2^{\mathbb{N}}$:

$$xE_0y \iff \exists m \in \mathbb{N} \quad \forall n > m \quad x(n) = y(n)$$

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$$id(1) <_B id(2) <_B \ldots <_B id(\mathbb{N}) <_B id(\mathbb{R}) <_B E_0$$

Countable structures

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Then each relation $R_i \in \mathcal{L}$ is a subset of $\mathbb{N}^{a(i)}$ where $a(i)$ is the parity of R_i.

Thus, $\mathcal{M}od(\mathcal{L})$ can be identified with $\prod_{i\in\mathbb{N}} 2^{\mathbb{N}^{a(i)}}$ or $2^{\mathbb{N}}$ and equip this space with the product topology to make $\mathcal{M}od(\mathcal{L})$ a standard Borel space.

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The logic action of S_∞ on $\mathcal{M}od(\mathcal{L})$ is given by

$$f \cdot M \models \phi(a_1, \ldots, a_n) \iff M \models \phi(f^{-1}(a_1), \ldots, f^{-1}(a_n)).$$

The orbit equivalence relation of S_∞ on $\mathcal{M}od(\mathcal{L})$ gives rise to the isomorphism equivalence relation.

Classification by countable structures

Definition

An equivalence relation E on a Borel space X is classifiable by countable structures if there is a countable language $\mathcal{L}$ and a Borel map $\phi\colon X \to \mathcal{M}od(\mathcal{L})$ such that

$$xEy \iff \phi(x) \cong \phi(y).$$

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The equivalence relations that are classifiable by countable structures include equivalence relations that can be reasonably classified using countable groups, graphs, fields, etc. as complete invariants.

The space of actions

Γ is a countable infinite group and (X, μ) a standard probability space (Borel isomorphic to [0, 1] with Lebesgue measure.)
$\Gamma \curvearrowright (X, \mu)$ by Borel automorphisms. This gives rise to the orbit equivalence relation

$$E_\Gamma = \{(x, \gamma \cdot x) \mid x \in X, \gamma \in \Gamma\}.$$

The action is:

- free if for any $\gamma \in \Gamma$, $\gamma \cdot x = x \implies \gamma = e$.
- measure preserving if for any Borel $A \subset X$ and $\gamma \in \Gamma$, $\mu(A) = \mu(\gamma \cdot A)$.
- ergodic if for any Borel Γ-invariant $A \subset X$, $\mu(A) = 1$ or $\mu(A) = 0$.

The space of actions

For a group Γ, $A(\Gamma, X, \mu)$ is the space of measure preserving actions of Γ on (X, μ) where (X, μ) is a standard probability space.

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$Aut(X, \mu)$ is a Polish space with the weak topology generated by the functions

$$A \mapsto T(A) \quad A \in MALG_{\mu}, T \in Aut(X, \mu)$$

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The space of homomorphisms of Γ into $Aut(X, \mu)$ is a closed subset of $Aut(X, \mu)^\Gamma$. The space of free and ergodic actions, which will be denoted $\mathcal{A}_\Gamma$ is closed in $A(\Gamma, X, \mu)$.

Orbit equivalence

Definition

Two actions $\Gamma \curvearrowright (X, \mu)$, $\Delta \curvearrowright (Y, \nu)$ are orbit equivalent if there are conull subsets $A \subset X, B \subset Y$ and a measure-preserving bijection $\phi \colon A \to B$ such that for all $x \in A$,

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Theorem (Feldman-Moore, 1977)

A measure space isomorphism $\phi\colon X \to Y$ extends to $L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Delta$ iff ϕ is an orbit equivalence.

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What can be said about the Borel complexity of OE_Γ?

Amenable groups

Theorem (Dye, about 1960)

Any two measure preserving ergodic actions of $\mathbb{Z}$ are orbit equivalent.

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Theorem (Ornstein-Weiss, 1980)

Any two measure preserving ergodic actions of an amenable group are orbit equivalent to such an action of $\mathbb{Z}$.

Almost invariant vectors

Let $\pi\colon \Gamma \to U(H)$ be a unitary representation of Γ on some Hilbert space H.

Then $v \in H$ is a π-invariant vector if for all $\gamma \in \Gamma$, $\pi(\gamma) \cdot v = v$.

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Definition

π admits almost invariant vectors if for any $Q \subset \Gamma$ finite, $\epsilon > 0$, there is a unit vector $v \in H$, such that

$$\forall \gamma \in Q \quad ||\pi(\gamma) \cdot v - v|| < \epsilon.$$

Amenable groups

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Equivalently, if for every $\epsilon > 0$, $A \subset \Gamma$ finite, there is a finite set $F \subset \Gamma$ such that

$$|\gamma \cdot A \Delta A| < \epsilon \cdot |A| \quad \forall \gamma \in A.$$

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Examples: finite groups, $\mathbb{Z}$, abelian groups

Non-example: $\mathbf{F}_2$

Non-amenable groups

Theorem (Connes-Weiss, Schmidt, 1980)

If Γ does not have property (T), then Γ admits at least 2 orbit inequivalent free, measure preserving, ergodic actions.

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Theorem (Bezuglyĭ - Golodets, 1981)

There is some countable infinite group that admits continuum many orbit inequivalent free, measure preserving, ergodic actions.

Property (T)

Definition

Γ has property (T) if there is a finite $Q \subset \Gamma$ and $\epsilon > 0$ such that for any unitary representation π of Γ, if π admits a (Q, ϵ)-invariant vector, then π admits an invariant unit vector.

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Theorem (Hjorth, 2005)

All groups with property (T) admit continuum many orbit inequivalent free, measure preserving, ergodic actions.

Relative property (T)

Definition

If $\Delta \leq \Gamma$, then the pair (Γ, Δ) has relative property (T) if there is a finite $Q \subset \Gamma$ and $\epsilon > 0$ such that for any unitary representation π of Γ, if π admits a (Q, ϵ)-invariant vector, then $\pi|\Delta$ admits an invariant unit vector.

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If we consider the usual action of $SL_2(\mathbb{Z})$ on $\mathbb{Z}^2$ by matrix multiplication, then the pair $(SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2, \mathbb{Z}^2)$ has relative property (T).

$\mathbf{F}_n$ embeds into $SL_2(\mathbb{Z})$ as a finite index subgroup to induce an action of $\mathbf{F}_n$ on $\mathbb{Z}^2$. The pair $(\mathbf{F}_n \ltimes \mathbb{Z}^2, \mathbb{Z}^2)$ also has relative property (T).

Relative property (T)

$SL_2(\mathbb{Z})$ also acts on $(\mathbb{T}^2, h)$ where h is the Haar measure.

We may identify $\mathbb{Z}^2$ with the group of characters on $\mathbb{T}^2$.

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Theorem (Gaboriau - Popa, 2006)

$\mathbf{F}_n$ for $n \geq 2$ admits continuum many orbit inequivalent free, measure preserving, ergodic actions.

Other classes of groups

The following classes of non-amenable groups also admit continuum many orbit inequivalent actions:

- Weakly rigid groups (Popa, 2006);
- Products of groups satisfying a certain cohomological property (Monod-Shalom, 2006);
- Mapping class groups (Kida, 2007).

Relative property (T) of $\mathbf{F}_2$

For $\Delta \leq \Gamma$ and an action $\Delta \curvearrowright (Z, \nu)$, one can induce an action of Γ on the space (Z^N, ν^N) where $N = [\Gamma : \Delta]$.

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Theorem (Ioana)

If $\mathbf{F}_2 \leq \Gamma$, then Γ admits continuum many orbit inequivalent free, measure preserving, ergodic actions.

There are non-amenable groups that don't contain a copy of $\mathbf{F}_2$ (Ol′šanskiĭ, 1980).

Extension from subgroups to subequivalence relations

Suppose $\Gamma \curvearrowright^{a_0} (X, \mu)$ and $\Delta \curvearrowright^{b_0} (X, \mu)$ such that $E_{\Delta}^{b_0} \subset E_{\Gamma}^{a_0}$.
Given $\Delta \curvearrowright^{a} (Z, \nu)$, we produce a way to induce actions $\Delta \curvearrowright^{c} (Y, m)$ and $\Gamma \curvearrowright^{d} (Y, m)$.

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Theorem (E.)

Suppose that there exist free, measure preserving actions $\Gamma \curvearrowright (X, \mu)$, $\mathbf{F}_2 \curvearrowright (X, \mu)$ such that Γ acts ergodically and $E_{\mathbf{F}_2} \subset E_\Gamma$. Then Γ admits continuum many orbit inequivalent free, measure preserving ergodic actions.

Extension from subgroups to subequivalence relations

Theorem (Gaboriau-Lyons)

Every non-amenable group admits a free, measure preserving, ergodic action whose orbit equivalence relation contains a subequivalence relation given by a free, measure preserving action of $\mathbf{F}_2$.

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Corollary

Every non-amenable group admits continuum many orbit inequivalent free, measure preserving, ergodic actions.

$\mathbf{F}_2$ admits continuum many non-isomorphic irreducible representations. These can be turned into actions of $\mathbf{F}_2 \curvearrowright (Z_i, \nu_i)$.

$\mathbf{F}_2$ also acts on $(\mathbb{T}^2, h)$ where h is the Haar measure. Then $\hat{\mathbb{T}}^2 \cong \mathbb{Z}^2$ and $(\mathbf{F}_2 \ltimes \mathbb{Z}^2, \mathbb{Z}^2)$ has relative property T.

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Using the actions of $\mathbf{F}_2$ and Γ from the theorem of Gaboriau and Lyons, we induce from the diagonal action $\mathbf{F}_2 \curvearrowright^a (Z_i \times \mathbb{T}^2, \nu_i \times h)$.

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Of these induced actions, only countably many will be orbit equivalent to each other.

Non-classification by countable structures

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E_0 can be reduced to OE_Γ and OE_Γ is not classifiable by countable structures:

- $\Gamma = \mathbf{F}_n$ or Γ has Property(T) (Tornquist);
- Γ has a copy of $\mathbf{F}_2$ (Ioana-Kechris);
- Γ is non-amenable (E-Ioana-Kechris-Tsankov).

OE_Γ is very complex. It is impossible to assign a real valued (or countable algebraic groups, etc) invariant to OE_Γ.

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Hjorth showed that the action of $U(H)$ by conjugacy on the irreducible unitary representations of $\mathbf{F}_2$ on H is turbulent.

Results about II_1 factors

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1. Isomorphism of factors is not classifiable by countable structures;
2. For any countable language $\mathcal{L}$, isomorphism on $\mathcal{M}od(\mathcal{L})$ is Borel reducible to isomorphism of factors;
3. Isomorphism of factors is Borel reducible to an equivalence relation arising from a continuous action of the unitary group of $l^2(\mathbb{N})$ on a Polish space. As a result, it is not the case that every analytic equivalence relation Borel reduces to isomorphism of factors.

Inducing an action of Γ

Let $\Gamma \curvearrowright^{a_0} (X, \mu)$ be free, measure preserving, ergodic and let $\Delta \curvearrowright^{b_0} (X, \mu)$, $\Delta \curvearrowright^{a} (Z, \nu)$ be free, measure preserving such that

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Definition

$$Y = \{(x, f) \mid f\colon [x]_\Gamma \to Z, \quad f(\gamma_0 \cdot x) = \gamma_0 \cdot f(x) \quad \forall \gamma_0 \in \Delta\}.$$

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We may assume that every E_Γ equivalence class contains infinitely many E_Δ equivalence classes (the number of equivalence classes is constant since E_Γ is ergodic).

Y will be represented as $(X \times Z^{\mathbb{N}}, \mu \times \nu^{\mathbb{N}})$.

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Given (x, f), f can be represented by choosing one value for each Δ-equivalence class in $[x]_\Gamma$.

There exists a Borel sequence of functions $\{g_i \colon X \to X\}_{i \in \mathbb{N}}$ such that

- $g_0(x) = x$ for every $x \in X$
- given $x \in X$, $\{g_i(x)\}_{i \in \mathbb{N}}$ enumerates a transversal for the Δ-equivalence classes in $[x]_\Gamma$;
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Then identify (x, f) with

$$(x, f(g_0(x)), f(g_1(x)), \ldots) \in (X \times Z^{\mathbb{N}}, \mu \times \nu^{\mathbb{N}}).$$

Inducing an action of Γ

Let $\Gamma \curvearrowright Y$ by

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This is given by a cocycle $\beta \colon \Gamma \times X \to S_\infty \ltimes \Delta^{\mathbb{N}}$.
So then $\Gamma \curvearrowright^c (X \times Z^{\mathbb{N}}, \mu \times \nu^{\mathbb{N}})$ by

$$\gamma \cdot (x, f) = (\gamma \cdot x, \beta(\gamma, x) \cdot f)$$

where
$S_\infty \ltimes \Delta^{\mathbb{N}} \curvearrowright Z^{\mathbb{N}}$ by

$$(\alpha, \delta) \cdot f(k) = \delta(k) \cdot f(\alpha^{-1}(k)).$$

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$$\sigma(\delta, x) = \gamma \iff \delta \cdot x = \gamma \cdot x.$$

We also get a free, measure preserving action $\Delta \curvearrowright^{d} (X \times Z^{\mathbb{N}}, \mu \times \nu^{\mathbb{N}})$ by letting

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We obtain a probability measure m on Y that is ergodic with respect to the action c of Γ by taking an ergodic decomposition.

Properties of construction

Lemma

Let $\mathbf{F}_2 \curvearrowright^{a} (\mathbb{T}^2, h)$ and $\mathbf{F}_2 \curvearrowright^{a_\pi} (Z, \nu)$ be free, measure preserving, weakly mixing. Then there are actions $\Gamma \curvearrowright^{c} (Y, m)$ and $\mathbf{F}_2 \curvearrowright^{d} (Y, m)$ such that:

1. $\Gamma \curvearrowright^{c} (Y, m)$ is free, measure preserving, ergodic;
2. $\mathbf{F}_2 \curvearrowright^{d} (Y, m)$ is free, measure preserving;
3. $E^{d}_{\mathbf{F}_2} \subset E^{c}_{\Gamma}$;
4. for any non non-null d-invariant subset $Y_0 \subset Y$, $a \times a_\pi$ is a factor of $d|Y_0$;
5. There is an $\mathbf{F}_2$-equivariant, measure preserving, surjective map $q\colon Y \to \mathbb{T}^2$ such that $\forall \gamma \in \Gamma \setminus \{e\}$,

$$m\big(\{y \in Y \mid q(\gamma^c \cdot y) = q(y)\}\big) = 0.$$

The Koopman representation

Definition

When $\Gamma \curvearrowright^{a} (X, \mu)$, the Koopman representation, κ_0^a, is given by

$$\gamma \cdot f(x) = f(\gamma^{-1} \cdot x) \quad \forall f \in L_0^2(X, \mu), \gamma \in \Gamma.$$

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For two actions $\Gamma \curvearrowright^a (X, \mu)$, $\Gamma \curvearrowright^b (Y, \nu)$, b is a factor of a if there is a measure preserving, surjective map $\phi \colon X \to Y$ such that

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f b is a factor of a, then $\kappa_0^b \leq \kappa_0^a$.
If b is conjugate to a, then $\kappa_0^b \cong \kappa_0^a$.

Proof

Let $\{\pi_i\}_{i\in I}$ be a set of continuum many non-equivalent, irreducible weakly mixing representations of $\mathbf{F}_2$. These can be turned into actions $\mathbf{F}_2 \curvearrowright^{a_{\pi_i}} (Z_i, \nu_i)$ such that

$$\pi_i \cong \pi_j \implies \kappa_0^{a_{\pi_i}} \cong \kappa_0^{a_{\pi_j}}$$

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For each $i \in I$, let c_i and d_i be actions of Γ and $\mathbf{F}_2$, respectively, that are given by the lemma.

Proof

Goal: Each c_i is only orbit equivalent to countably many c_j.

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(Ioana) There is an uncountable set $J_0 \subset J$ such that for all $j, k \in J_0$, there are non-null $\mathbf{F}_2$-invariant subsets $Y_j' \subset Y_j$, $Y_k' \subset Y_k$ such that $d_j | Y_j'$ is conjugate to $d_k | Y_k'$.

Proof

Let $i \in J_0$. Then for any $j \in J_0$,

$$\pi_j \leq \kappa_0^{a_{\pi_j}} \leq \kappa_0^{a \times a_{\pi_j}}.$$

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Since $\kappa_0^{d_i | Y_i'} \leq \kappa_0^{d_i}$,

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However, a separable representation may only contain countably many non-equivalent irreducible representations. So each c_i can only be orbit equivalent to countably many c_j.