Singular Cardinals in the 20th Century

Menachem Kojman

July 2009

König's Heidelberg scandal, 1904

Singular cardinals appeared on the mathematical world stage two years before they were defined. At the third international congress of mathematicians in Heidelberg, on August 10, 1904, Julius König refuted Georg Cantor's two major conjectures simultaneously, when proving that the continuum could not be ordered in one of the order-types on Cantor's list of $\aleph$-s. König's proof went as follows: suppose that the continuum is some $\aleph_\beta$. Look at the sequence $\aleph_{\beta+1}, \aleph_{\beta+2}, \ldots, \aleph_{\beta+n}, \ldots$ for all natural n. Its limit, $\aleph_{\beta+\omega} = \sum_n \aleph_{\beta+n} < \prod_n \aleph_{\beta+n} \leq (\aleph_{\beta+\omega})^{\aleph_0}$ but by a lemma from Bernstein's thesis, $(\aleph_{\beta+\omega})^{\aleph_0} = 2^{\aleph_0}\aleph_{\beta+\omega} = \aleph_{\beta+\omega}$.

Hausdorff's letter to Hilbert

After the continuum problem plagued me in Wengen almost like an obsession, my first look here was naturally directed to Bernstein's dissertation. The bug lay exactly in the expected place. on Page 50 [. . .] Berstein's consideration employs a recursion . . . which does not hold for such alephs that have no immediate predecessor, that is, exactly those alephs which König required. I had written to this effect to Mr. König while still on the road and explained this as far as I could without using Bernstein's work, but so far received no reply. Now I am yet even more convinced that König's proof is wrong and I believe König's theorem to be the top of impossibility. On the other hand, I am sure you also do not believe that Mr. Cantor discovered during this last few weeks what he has been looking for in vain for the last 30 years. Thus, your problem number 1 seems to remain after the Heidelberg congress exactly where you left it after the Paris congress.

Advanced König's inequalities

Suppose that A is a set of regular cardinals with $|A| < \min A$. For every ideal I over A let $<_I$ be defined on $\prod A$. $(\prod A, <_I)$ is a quasi ordered set with no maximum, so has a bounding number $\mathfrak{b}(\prod A, <_I)$. By pcf A the set of all (regular) bounding numbers is denoted. There is a sequence of **generators** $\langle B_\lambda : \lambda \in \text{pcf } A \rangle$ which form a basis for all reduced products of A.

- $\lambda = \text{tcf}(\prod A, <_{\langle \{B_\theta : \theta < \lambda\} \rangle})$.
- $\mathfrak{b}(\prod A, <_I) = \lambda \iff \lambda$ is the least such that $B_\lambda \notin I$.

Relation to arithmetic

The Binomial $\binom{\mu}{\mathrm{cf}\,\mu} := \mathrm{cf}([\mu]^{\mathrm{cf}\,\mu}, \subseteq)$. And if $\mu = \aleph_\omega$, this is $\max \mathrm{pcf}\{\aleph_n\}$. Which is smaller than $\aleph_{\omega_4}$.
Gitik proved that at the first fixed point of the $\aleph$-function this relation breaks — there is no bound on the binomial of the first fixed point
Note that the size of the continuum is not so related to $\binom{\aleph_\omega}{\omega}$ — the continuum is allowed to be much bigger that $\aleph_{\omega_4}$. e.g. be real valued measurable.

Hausdorff

Felix Hausdorff began working in set theory in 1902. He was the one who discovered König's mistake. And who defined cofinality, regular and singular cardinals in 1906. The name singulär was Borrowed from König. Hausdorff proved from the GCH: in every infinite cardinalily there is a universal linearly ordered set. Hausdorff studies **pantachies** — linearly ordered subsets of $(\omega^\omega, <^*)$. Hausdorff constructed a hausdorff gap! (and published it again after about 30 years).
Rothberger proved that $\mathfrak{b}(\omega^\omega, <^*)$ was regular, considered $\mathfrak{d}$. By not proving that $\mathfrak{d}$ was regular, he pointed to the possibility that $\mathfrak{d}$ could be singular.

Eubs and the trichotomy

All increasing sequences in $(\mathrm{On}^\omega, <_I)$ of cofinality larger than continuum have eubs.

The trichotomy says, basically, that unless a $<_I$ increasing sequence is caught up in a copy of ω^ω, it will have an eub.

Every increasing unbounded sequence in $(\prod A, <_I)$ is dominated by a exact sequences in the same product.

Here the verifiability of $V = L$ is extremely evident — in the use of $I[\lambda]$ and club guessing.

Alexandrov and Uryson, the 1920

Generalization of (countable) compactness — what we now call compactness.

- For all infinite regular cardinal λ, X has the λ-CAP property.
- Every infinite open cover has a finite subcover.

Why not mention the singulars?

$$(\mu = \sum_{\alpha<\kappa} \kappa_\alpha \wedge \bigwedge_\alpha \kappa_\alpha - CAP) \Longrightarrow \mu - CAP.$$

Miscenko 1959: there are spaces that satisfy λ-CAP for regulars but not for singulars. Linearly Lindelöf, not Lindelöf M. E. Rudin 1970: a Dowker space in ZFC.

$$X^R = \{f : (\exists m)(\forall n > 0)\aleph_0 < \mathrm{cf}\, f(n) \leq \omega_m\}.$$

P. Simon 1970: some Baire measures do not extend to regular Borel measures.

Dowker spaces from pcf below the continuum

Theorem (K., H. Michalewski)

Below the first real valued measurable cardinal, every Baire measure in a Rudin-type space extends to a Borel measure.

One can construct a Rudin type space of cardinality $\aleph_{\omega+1}$ from pcf scales without ever taking a single infinite product.

Arhangelskii, Juhasz and Szentmiklossy

A space X is uncountably compact if it satisfies λ-CAP for all uncountable λ. Such a space is more than Lindelöf. Linearly Lindelöf is less that Lindelöf.

Theorem (Arkhangelskii 2008)

A regular uncountably compact space is $\aleph_\omega$ concentrated on a compact subspace.

Theorem (Juhasz and Szentmiklossy 2009)

If X is linearly Lindelöf and $\aleph_\omega$ compact then it is uncountably Lindelöf (hence Lindelöf).

Whitehead groups and freeness

An abelian group A is **Whitehead** iff whenever $0 \to \mathbb{Z} \to B \to A \to 0$ is exact then the arrow $B \to A$ can be "lifted".

Free (abelian) groups are W. and countable W. groups are free.

MA($\aleph_1$) implies the existence of a non-free W. group.

V=L implies all W. groups are free.

Induction. At singular, no constructibility is needed! L predicts the universe correctly — at singulars, at least.

Shelah's compactness of singulars.

Annus Mirabilis — 1974

After discovering an elementary proof of Silver's theorem, Jensen proved the covering Lemma. At singulars, the connections between L and V became clear. Silver's theorem follows from

$$\mathrm{tcf}(\prod_{\alpha<\omega_1} \aleph_{\alpha+1}, <_{NS}) = \aleph_{\omega_1+1}.$$

Metamathematics, ultraproducts, forcing, generic ultraproducts, large ideals — all were employed to obtain Silver's theorem before elementary proofs arrived.

GCH in ZFC

$\binom{\lambda}{\kappa}^*$ — least size of $\mathcal{F} \subseteq [\lambda]^\kappa$ such that for every $X \in \lambda]^\kappa$ there is $A \in [\mathcal{F}]^{<\kappa}$ such that $X \subseteq \bigcup A$.

Theorem

for every $\lambda \geq \beth_\omega$ for all but boundedly many $\kappa < \beth_\omega$ it holds that $\binom{\lambda}{\kappa}^* = \lambda$.

Corollary: above $\beth_\omega$ whenever GCH holds there is a diamond.

Latest: for every $\lambda \geq \omega_1$ if $2^\lambda = \lambda^+$ then $\diamondsuit(\lambda^+)$.

Problem (a converse to Hausdorff): is a singular strong limit if and only if it carries a universal linear ordering? True below the least fixed point of second order.

21-st Century

Singular cardinals keep working for us.

Viale, Rinot, Spadaro and Milovich, Juhasz-Szentmiklossy, Soukup — uses of the elementary theory of singulars from the last few weeks.

Vote: composites or accessible limits?