Rudimentary recursion and provident sets

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The Σ_1 recursion theorem of Kripke-Platek set theory KP proves for G a Σ_1 function that if G is total, so is the function F given by the recursion

$$F(x) = G(F \restriction x),$$

and further F is provably equal to a Σ_1 function.

If the defining function G is rudimentary in the sense of Jensen, we shall speak of F as given by a *rudimentary recursion*, or, more briefly, that F *is rud rec*.

In favourable cases we may also use this terminology when F is intended to be a function defined on On rather than on V, or defined by recursion on other well-founded relations related to the epsilon relation.

A set of generators for the class of rudimentary functions

$$R_0(x, y) = \{x, y\}$$
$$R_1(x, y) = x \setminus y$$
$$R_2(x) = \bigcup x$$
$$R_3(x) = \mathrm{Dom}\,(x)$$
$$R_4(x, y) = x \times y$$
$$R_5(x) = x \cap \{(a, b)_2 \mid_{a,b} a \in b\}$$
$$R_6(x) = \{(b, a, c)_3 \mid_{a,b,c} (a, b, c)_3 \in x\}$$
$$R_7(x) = \{(b, c, a)_3 \mid_{a,b,c} (a, b, c)_3 \in x\}$$
$$A_{14}(x, y) = x``\{y\}\,[= \mathrm{Dom}\,((x \cap ([\bigcup\bigcup x] \times \{y\}))^{-1})]$$
$$R_8(x, y) = \{x``\{w\} \mid_w w \in y\}$$

Some rudimentary recursions

EXAMPLE The definition of *rank*:

$$\varrho(x) = \bigcup\{\varrho(y) + 1 \mid_y y \in x\}$$

EXAMPLE The definition of *transitive closure*:

$$\mathrm{tcl}(x) = x \cup \bigcup\{\mathrm{tcl}(y) \mid_y y \in x\}$$

EXAMPLE Let $\mathcal{S}(x)$ be the set of finite subsets of x. *Restricted to ordinals*, this has a rudimentarily recursive definition:

$$\mathcal{S}(0) = \{\varnothing\};\ \mathcal{S}(\zeta{+}1) = \mathcal{S}(\zeta)\cup\{x\cup\{\zeta\} \mid_x x \in \mathcal{S}(\zeta)\};\ \mathcal{S}(\lambda) = \bigcup_{\nu<\lambda} \mathcal{S}(\nu).$$

Restricted to the hereditarily finite sets, it hasn't.

$$
\begin{aligned}
\mathbb{T}(u) \;=_{\mathrm{df}}\; & u \cup \{u\} \cup [u]^1 \cup [u]^2 \\
& \cup \{x \smallsetminus y \;|_{x,y}\; x, y \in u\} \\
& \cup \left\{\bigcup x \;\Big|_{x}\; x \in u\right\} \\
& \cup \left\{\mathrm{Dom}\,(x) \;\Big|_{x}\; x \in u\right\} \\
& \cup \{u \cap (x \times y) \;|_{x,y}\; x, y \in u\} \\
& \cup \left\{x \cap \{(a,b)_2 \;|_{a,b}\; a \in b\} \;\Big|_{x}\; x \in u\right\} \\
& \cup \left\{u \cap \{(b,a,c)_3 \;|_{a,b,c}\; (a,b,c)_3 \in x\} \;\Big|_{x}\; x \in u\right\} \\
& \cup \left\{u \cap \{(b,c,a)_3 \;|_{a,b,c}\; (a,b,c)_3 \in x\} \;\Big|_{x}\; x \in u\right\} \\
& \cup \left\{x``\{w\} \;\Big|_{x,w}\; x \in u, w \in u\right\} \\
& \cup \left\{u \cap \{x``\{w\} \;\big|_{w}\; w \in y\} \;\Bigg|_{x,y}\; x, y \in u\right\}.
\end{aligned}
$$

Proposition $\mathbb{T}$ *is rudimentary,* $u \subseteq \mathbb{T}(u)$ *and* $u \in \mathbb{T}(u)$*. If* u *is transitive, then* $\mathbb{T}(u)$ *is a set of subsets of* u*,* $\mathbb{T}(u)$ *is transitive,* $\mathit{rank}(\mathbb{T}(u)) = \mathit{rank}(u) + 1$*, and* $\bigcup_{n\in\omega} \mathbb{T}^n(u)$ *is the rud closure of* $u \cup \{u\}$*.*

Remark Recursively define $T(x) = \bigcup_{y\in x} \mathbb{T}(T(y))$; then $T(x)$ always equals $T_{\varrho(x)}$, where

$$T_0 = \varnothing; \qquad T_{\nu+1} = \mathbb{T}(T_\nu); \qquad T_\lambda = \bigcup_{\nu<\lambda} T_\nu$$

which can be said in one breath as

$$T_\zeta = \bigcup_{\nu<\zeta} \mathbb{T}(T_\nu)$$

then $L = \bigcup_{\nu\in ON} T_\nu$, and $J_\nu = T_{\omega\nu}$; but $\nu \mapsto \omega\nu$ is not rud rec.

Proposition *If $F(\vec{x})$ is a rudimentary function of several variables, there is an $\ell \in \omega$ such that for all transitive u, if each argument in $\vec{x}$ is in u, then $F(\vec{x}) \in \mathbb{T}^\ell(u)$.*

Proof : The stated property holds of the nine generating functions and is preserved under composition. $\dashv$

Corollary (Gandy; Jensen) *If F is rudimentary, then there is a finite ℓ such that the rank of the value is at most the maximum of the ranks of the arguments, plus ℓ.*

Proof : the function $\mathbb{T}$ increases rank by exactly 1. $\dashv$

Rudimentary recursion from parameters

Let p be a set. We call a unary function F *p-rud rec* if there is a binary rudimentary G such that for all x,

$$F(x) = G(p, F \restriction x).$$

EXAMPLE Ordinal addition is given by the recursion

$$A(\alpha, 0) = \alpha;\ A(\alpha, \beta + 1) = A(\alpha, \beta) + 1;\ A(\alpha, \lambda) = \bigcup_{\nu<\lambda} A(\alpha, \nu)$$

For each α that is an α-rud recursion on the second variable β.

Provident sets

DEFINITION A set A is *p-provident*, where p is a set, if it is non-empty, transitive, closed under pairing and for all p-rud rec F and all x in A, $F(x) \in A$.

REMARK If A is p-provident, $p \in A$.

EXAMPLE The Jensen fragment J_ν is $\varnothing$-provident for all $\nu \geqslant 1$.

DEFINITION A is *provident* if it is p-provident for every $p \in A$.

EXAMPLE Each J_{ω^ν} is provident.

REMARK For provident sets, it is unnecessary to demand that they be closed under pairing, for if $x \in A$, the function $y \mapsto \{x, y\}$ is x-rud rec, being given by the recursion $F(y) = \{x, \mathrm{Dom}\, F \restriction y\}$.

Bounding rudimentary functions in a finite progress

DEFINITION A ξ-*progress* is a sequence $\langle P_\nu \mid \nu \leqslant \xi \rangle$ of transitive sets such that for each $\nu < \xi$, $\mathbb{T}(P_\nu) \subseteq P_{\nu+1}$ and for each limit ordinal $\lambda \leqslant \xi$, $\bigcup_{\nu<\lambda} P_\nu \subseteq P_\lambda$.

The progress is *strict* if for each $\nu < \xi$, $P_{\nu+1} \subseteq \mathcal{P}(P_\nu)$; and *continuous* if for each limit $\lambda \leqslant \xi$, $P_\lambda = \bigcup_{\nu<\lambda} P_\nu$.

THEOREM *Let R be a rudimentary function of n variables. There is a $c_R \in \omega$ such that for every c_R-*progress $P_0, P_1, \ldots, P_{c_R}$,

$$R``P_0^n \subseteq P_{c_R}.$$

DEFINITION We call c_R the *rudimentary constant* of R.

The canonical progress towards a given transitive set

Let c be a transitive set. Let $c_\zeta = c \cap \{x \mid \varrho(x) < \zeta\}$. Since c is transitive, $c_{\zeta+1}$ will be a set of subsets of c_ζ; in fact $c_{\zeta+1} = c \cap \{x \mid x \subseteq c_\zeta\}$; we shall use this as a direct recursive definition below.

If $c_{\zeta+1} = c_\zeta$, then $c_\zeta = c$ and for all $\xi > \zeta$, $c_\xi = c_\zeta$; so that that first happens when $\zeta = \varrho(c)$.

Using c as a parameter we define a sequence of pairs $((c_\nu, P^c_\nu))_\nu$ by a rud recursion on ν. Each P^c_ν will be of rank ν; we shall use the function $\mathbb{T}$, but we shall also "feed" stages of c into the process.

Definition

$$\begin{array}{lllllllll} c_0 & = & \varnothing & c_{\nu+1} & = & c \cap \{x \mid x \subseteq c_\nu\} & c_\lambda & = & \bigcup_{\nu<\lambda} c_\nu \\ P^c_0 & = & \varnothing & P^c_{\nu+1} & = & \mathbb{T}(P^c_\nu) \cup \{c_\nu\} \cup c_{\nu+1} & P^c_\lambda & = & \bigcup_{\nu<\lambda} P^c_\nu \end{array}$$

Lemma *Each P^c_ν is transitive. $P^c_\nu \subseteq P^c_{\nu+1}$. $P^c_\nu \in P^c_{\nu+1}$; and so for $\nu < \zeta$, $P^c_\nu \subseteq P^c_\zeta$ and $P^c_\nu \in P^c_\zeta$.*

Remark $c_\nu = c \cap P^c_\nu$; $\varrho(P^c_\nu) = \nu$.

Remark P^c_ν may be defined by a single rud recursion on ordinals:

$$P^c_0 = \varnothing;\ P^c_{\nu+1} = \mathbb{T}(P^c_\nu) \cup \{c \cap P^c_\nu\} \cup (c \cap \{x \mid x \subseteq P^c_\nu\});\ P^c_\lambda = \bigcup_{\nu<\lambda} P^c_\nu.$$

Remark Each P^c_λ is rud closed, for λ a limit ordinal; $P^c_\omega = V_\omega$.

Two important properties of p-rud rec functions.

THE DEFINABILITY LEMMA *Let F be p-rud recursive, given by G. Then "f is an F-attempt" is a Δ_0 predicate of p and f.*

REMARK If F is rud rec (in a parameter), so is $x \mapsto F \restriction x$ (in the same parameter).

THE PROPAGATION LEMMA *Let G be a binary rudimentary function. Then there is a ternary rudimentary function H_G, obtainable uniformly from G, such that for any set p, if F be the p-rud rec function given by the recursion $F(x) = G(p, F \restriction x)$, and if P^+ and P be transitive sets with $P \subseteq P^+ \subseteq \mathcal{P}(P)$, then*

$$F \restriction P^+ = H_G(p, F \restriction P, P^+).$$

Proof : If $x \in P^+$, then $x \subseteq P$, so $F \restriction x = (F \restriction P) \restriction x$ so $F(x) = G(p, (F \restriction P) \restriction x)$. Hence

$$F \restriction P^+ = \{(G(p, (F \restriction P) \restriction x), x)_2 \mid_x x \in P^+\}.$$

We take $H_G(p, f, q) \equiv \{(G(p, f \restriction x), x)_2 \mid_x x \in q\}$. ⊣

A new symbol

DEFINITION $F \upharpoonright\!\!\upharpoonright u =_{\text{df}} \{F\upharpoonright x \mid x \in u\}$

Bounding rudimentarily recursive functions in a single canonical progress

THEOREM *Let F be p-rud rec, given by G. Then there exist s_F and g_F in ω such that for any transitive c and any ordinal ν_0 with $p \in P^c_{\nu_0}$, any non-successor ordinal λ and any $k \in \omega$,*

$$\textit{(i)}\ F \upharpoonright\!\!\upharpoonright P^c_\lambda \subseteq P^c_{\nu_0+\lambda}; \qquad \textit{(ii)}\ F\upharpoonright P^c_{\lambda+k} \in P^c_{\nu_0+\lambda+s_F+k\cdot g_F}.$$

THEOREM *Let θ be indecomposable and c a transitive set. Then P^c_θ is provident.*

Proof : Let $p \in P^c_\theta$; choose $\nu_0 < \theta$ with $p \in P^c_{\nu_0}$. Let F be p-rud rec. Then for each limit $\eta < \theta$, $F \upharpoonright P^c_\eta \subseteq P^c_{\nu_0+\eta} \subseteq P^c_\theta$. So $F \upharpoonright P^c_\theta \subseteq P^c_\theta$, as required. $\dashv$

PROPOSITION *Let c be a transitive set and θ an indecomposable ordinal. Then*

$$P^c_\theta = P^{c_\theta}_\theta = \bigcup_{\lambda<\theta} P^{c_\lambda}_\theta.$$

Proposition *If θ is an indecomposable ordinal and C is a set of transitive sets such that any two members of C are members of a third, then $B =_{\rm df} \bigcup_{c \in C} P^c_\theta$ is provident. More generally, the union of a directed system of provident sets is provident.*

Proof : Given a parameter p in B and an argument x in B, choose $c \in C$ with both p and x in P^c_θ. We know that P^c_θ is provident, and so if F is p-rud rec, $F(x)$ is in P^c_θ and therefore in B. $\dashv$ (20·-1)

PROPOSITION *Let A be a provident set, and write $\theta(A)$ for the least ordinal not in A.*

A is rud closed;

A contains the rank $\varrho(x)$ of each member x of A;

A contains the transitive closure of each of its members;

$\theta(A)$ is indecomposable;

$\theta(A) = \varrho(A)$;

$A = \bigcup\{P^d_{\theta(A)} |_d \bigcup d \subseteq d \in A\}$

PROPOSITION *Let θ be an indecomposable ordinal, and let $(Q_\nu)_{\nu \leqslant \theta}$ be a θ-progress with $Q_\theta = \bigcup_{\nu<\theta} Q_\nu$. Then Q_θ is provident.*

Provident levels of the Jensen and Gödel hierarchies

PROPOSITION *If u is transitive and $\varnothing$-provident then so is rud(u).*

Proof : We take $P_n = \mathbb{T}^n(u)$, and $P_\omega = \bigcup_n P_n$. $\langle P_\nu \mid \nu \leqslant \omega \rangle$ is then a strict continuous ω-progress, so we may apply a previous proposition with $p = \varnothing$. $\dashv$

COROLLARY *Each non-empty J_ν is $\varnothing$-provident,*

THEOREM *J_ν is provident iff $\omega\nu$ is indecomposable. More generally, if c is a transitive set, $J_\nu(c)$ will be provident iff $\omega\nu$ is indecomposable and strictly greater than the rank of c.*

REMARK We need $\omega\nu$ to exceed the rank of c, as provident sets contain the ranks of their members.

Remark So although for a given p in L we must go to the first indecomposable ordinal above the moment of construction of p to find a J_ν which is p-provident, every subsequent J_ξ will also be p-provident.

Proposition *J_ω is provident. The next one will be J_{ω^2}.*

Proposition *Each L_λ is $\varnothing$-provident for limit λ.*

Proposition *L_λ is provident iff λ is indecomposable.*

$\mathcal{S}$-logic in provident sets

Proposition *Let A be provident; let $a \in A$. Then $\mathcal{S}(a) \in A$.*

Rudimentary recursion and forcing

Example Suppose we are making a forcing extension using a notion of forcing $\mathbb{P}$ that is a set of the ground model, assumed transitive. In the theory of forcing, a member y of the ground model is represented by the term $\hat{y}$ of the language of forcing, given by the recursion

$$\hat{y} =_{\mathrm{df}} \{(\mathbb{1}^{\mathbb{P}}, \hat{x}) \mid_x x \in y\}$$

This is a rudimentary recursion in a parameter, being of the form

$$F(a) = G(\mathbb{1}^{\mathbb{P}}, F \restriction a)$$

where G is the rudimentary function $(\mathbb{1}^{\mathbb{P}}, a) \mapsto \{\mathbb{1}^{\mathbb{P}}\} \times \mathrm{Im}(a)$.

Example If $\mathcal{G}$ is a generic filter on a notion of forcing $\mathbb{P}$ in a transitive model M, and we follow Shoenfield in treating all members of M as $\mathbb{P}$-names, the function $\mathrm{val}_{\mathcal{G}}(\cdot)$ defined for $a \in M$ is given by a rudimentary recursion with $\mathcal{G}$ as a parameter.

$$\mathrm{val}_{\mathcal{G}}(b) \;=_{\mathrm{df}}\; \{\mathrm{val}_{\mathcal{G}}(a) \mid_a \exists p{:}{\in}\mathcal{G}\ (p,\, a) \in b\}$$

The generic extension $M[\mathcal{G}]$ is then be defined as $\{\mathrm{val}_{\mathcal{G}}(a) \mid_a a \in M\}$.

Remark Note that the definition of the forcing relation $\Vdash$ has not been invoked in making these definitions, but its properties would be needed to show that $M[\mathcal{G}]$ has interesting properties.

Forcing in provident sets

DEFINITION $p \Vdash_0 \underline{a} \epsilon \underline{b} \iff_{\mathrm{df}} (p, a) \in b$.

$\Vdash_0$ is our first approximation to the relation $\Vdash$.

LEMMA *If* $p \Vdash_0 \underline{a} \epsilon \underline{b}$ *then* $a \in \bigcup\bigcup b$.

DEFINITION In future we shall write $\bigcup^2 x$ for $\bigcup\bigcup x$.

DEFINITION $p \Vdash_1 \underline{a} \epsilon \underline{b} \iff_{\mathrm{df}} \exists q {:\in} \bigcup^2 b\, [q \geq p \ \&\ (q, a) \in b]$.

LEMMA *If* $p \Vdash_1 \underline{a} \epsilon \underline{b}$ *and* $r \leq p$ *then* $r \Vdash_1 \underline{a} \epsilon \underline{b}$.

DEFINITION $p \Vdash \underline{b} = \underline{c} \iff_{\mathrm{df}}$

$$\forall\beta{:}{\in}\bigcup{}^2 b\ \forall r{:}{\le} p\left[r \Vdash_1 \underline{\beta}\ \epsilon\ \underline{b} \Rightarrow \exists t{:}{\le} r\ \exists\gamma{:}{\in}\bigcup{}^2 c\,(t \Vdash \underline{\beta} = \underline{\gamma}\ \&\ t \Vdash_1 \underline{\gamma}\ \epsilon\ \underline{c})\right]$$
$$\forall\gamma{:}{\in}\bigcup{}^2 c\ \forall r{:}{\le} p\left[r \Vdash_1 \underline{\gamma}\ \epsilon\ \underline{c} \Rightarrow \exists t{:}{\le} r\ \exists\beta{:}{\in}\bigcup{}^2 b\,(t \Vdash \underline{\gamma} = \underline{\beta}\ \&\ t \Vdash_1 \underline{\beta}\ \epsilon\ \underline{b})\right]$$

DEFINITION Let $\chi_{=}(p, b, c)$ be the characteristic function of the relation $p \Vdash^{\mathbb{P}} \underline{b} = \underline{c}$, so that it takes the value 1 if $p \Vdash^{\mathbb{P}} \underline{b} = \underline{c}$ and 0 otherwise.

The graph of $\chi_{=}$ on transitive sets is given by a $\mathbb{P}$-rudimentary recursion.

THE DEFINABILITY LEMMA "f is a $\chi_{=}$ attempt" is $\Delta_0(\mathbb{P}, f)$.

THE PROPAGATION LEMMA *Let* $F(u) = \chi_{=} \restriction (\mathbb{P} \times u \times u)$. *There is a rudimentary function* $H_{=}$ *such that for any transitive* P, *if* $P \subseteq P^+ \subseteq \mathcal{P}(P)$,

$$F(P^+) = H_{=}(\mathbb{P}, F(P), P^+).$$

Propagation of $\chi_{=}$

We have defined the progress P_ν^c for c a transitive set. We could continue to work with progresses of the above kind, but a problem would then arise at the end of the paper, in the proof that a set-generic extension of a provident set is provident.

Here it is better to work with other progresses, which might be called construction from e as a set and $\chi_{=}$ as a predicate, with the definition of $\chi_{=}$ evolving during the construction.

Definition Let e be a transitive set of which $\mathbb{P}$ is a member; let $\eta = \varrho(\mathbb{P})$. We define by a p-rudimentary recursion a sequence $((e_\nu, P_\nu^{e;=}, \chi_\nu^e)_3)_\nu$ of triples, thus obtaining a new progress $(P_\nu^{e;=})_\nu$. For every ν, e_ν will be defined as before; for $\nu \leqslant \eta$ we set $P_\nu^{e;=} =$

P^e_ν; for $\nu < \eta$, we set $\chi^e_\nu = \varnothing$ but at η, we set $\chi^e_\eta = \chi_{=} \restriction P^e_\eta$, which will be a set by the last Corollary. Thereafter we set

$$\begin{aligned} e_{\nu+1} &= e \cap \{x \mid x \subseteq e_\nu\} & e_\lambda &= \bigcup_{\nu<\lambda} e_\nu \\ P^{e;=}_{\nu+1} &= \mathbb{T}(P^{e;=}_\nu) \cup \{e_\nu\} \cup e_{\nu+1} \cup \{\chi^e_\nu \cap P^{e;=}_\nu\} & P^{e;=}_\lambda &= \bigcup_{\nu<\lambda} P^{e;=}_\nu \\ \chi^e_{\nu+1} &= H_{=}(\mathbb{P}, \chi^e_\nu, P^{e;=}_{\nu+1}) & \chi^e_\lambda &= \bigcup_{\nu<\lambda} \chi^e_\nu \end{aligned}$$

PROPOSITION *Let e be transitive, with $\mathbb{P} \in e$, and let θ be indecomposable and strictly greater than $\varrho(\mathbb{P})$. Then $P^{e;=}_\theta = P^e_\theta$.*

This reconstruction of P^e_θ shortens the delay for most χ^e_ν:

PROPOSITION *For any ordinal $\nu \geqslant \eta$, any limit ordinal $\lambda > \eta$ and $k \in \omega$,*

$$\chi^e_\nu = \chi_{=} \restriction P^{e;=}_\nu;$$
$$\chi^e_\nu \subseteq P^{e;=}_{\nu+6};$$
$$\chi^e_\lambda \subseteq P^{e;=}_\lambda;$$
$$\chi_{=} \restriction P^{e;=}_\nu \in P^{e;=}_{\nu+12}.$$

Propagation of χ_ϵ

We may now define $p \Vdash \underline{a} \epsilon \underline{b}$.

Definition $p \Vdash \underline{a} \epsilon \underline{b} \iff_{\mathrm{df}} \forall s :\leq p\ \exists t :\leq s\ \exists \beta :\in \bigcup^2 b\ \left[t \Vdash \underline{\beta} = \underline{a}\ \&\ t \Vdash_1 \underline{\beta} \epsilon \underline{b}\right]$.

Remark This is not a definition by recursion: indeed it is visibly rudimentary in $p \Vdash \underline{b} = \underline{c}$.

Definition Let $\chi_\epsilon(p, a, b)$ be the characteristic function of the relation $p \Vdash^{\mathbb{P}} \underline{a} \epsilon \underline{b}$.

Proposition *There is a natural number* s_ϵ *such that for each ordinal* $\nu \geqslant \eta$, $\chi_\epsilon \restriction P_\nu^{e;=} \in P_{\nu+s_\epsilon}^{e;=}$.

Construction of Cohen terms of affine delay for rudimentary functions

THEOREM *Let R be a rudimentary function of some number of arguments. Then there is a function $R^{\mathbb{P}}$, of the same number of arguments, with the property that if A is a provident set and $\mathbb{P} \in A$ a notion of forcing, then A is closed under $R^{\mathbb{P}}$ and, further, if $\mathcal{G}$ is an $(A, \mathbb{P})$-generic, then (to take the case of a function of two variables) for all x and y in A,* $\mathrm{val}_{\mathcal{G}}(R^{\mathbb{P}}(x, y)) = R(\mathrm{val}_{\mathcal{G}}(x), \mathrm{val}_{\mathcal{G}}(y))$.

COROLLARY *Let A be provident, $\mathbb{P} \in A$ and $\mathcal{G}$ $(A, \mathbb{P})$-generic. Then $A[\mathcal{G}]$ is rud closed and so a model of GJ_0.*

Propagation of Cohen terms for rud functions

PROPOSITION *Let R be a rudimentary function of some number of arguments, and let $R^{\mathbb{P}}$ be the corresponding function of names that we have defined. There is a natural number s_R such that whenever e is a transitive set with $\mathbb{P} \in e$, and ν is an ordinal not less than $\varrho(\mathbb{P})$,*

$$R^{\mathbb{P}} \restriction P_{\nu}^{e;\,=} \in P_{\nu+s_R}^{e;\,=}.$$

No new ordinals !

REMARK In section 6 of *The strength of Mac Lane set theory*, a forcing contruction is done over a non-standard model $\mathfrak{N}$, and it was there blithely stated without proof that the generic extension would bring no new “ordinals” Fortunately the model $\mathfrak{N}$ was power-admissible, and therefore certainly a model of PROVI, which is a sub-theory of KP, so that the present remarks justify that blithe statement; that is reassuring in view of the somewhat pathological models presented elsewhere.

Construction of rudimentarily recursive Cohen terms for rank and transitive closure

Rank and transitive closure are pure rud rec; we show here that $\mathbb{P}$-rud rec Cohen terms exist for them.

Let $S(\cdot)$ be the basic function $z \mapsto z \cup \{z\}$.

LEMMA *There is a rud function* $S^{\mathbb{P}}(\cdot)$ *such that* $\mathrm{val}_{\mathcal{G}}(S^{\mathbb{P}}(x)) = S(\mathrm{val}_{\mathcal{G}}(x))$.

DEFINITION $\varrho^{\mathbb{P}}(x) =_{\mathrm{df}} \bigcup^{\mathbb{P}}\{(p, S^{\mathbb{P}}(\varrho^{\mathbb{P}}(y)) \mid (p, y) \in x \,\&\, p \in \mathbf{P}\}$

REMARK $\varrho^{\mathbb{P}}$ is rud rec in the parameter $\mathbb{P}$.

LEMMA *Let A be provident, and $\mathbb{P} \in A$. For all $x \in A$,* $\mathrm{val}_{\mathcal{G}}(\varrho^{\mathbb{P}}(x)) = \varrho(\mathrm{val}_{\mathcal{G}}(x))$.

DEFINITION $\mathrm{tcl}^{\mathbb{P}}(x) =_{\mathrm{df}} x \cup^{\mathbb{P}} \bigcup^{\mathbb{P}}(\{(p, \mathrm{tcl}^{\mathbb{P}}(z)) \mid (p, z) \in x\})$.

REMARK $\mathrm{tcl}^{\mathbb{P}}$ is rud rec in the parameter $\mathbb{P}$.

LEMMA *Let A be provident, and $\mathbb{P} \in A$. For all $x \in A$,* $\mathrm{val}_{\mathcal{G}}(\mathrm{tcl}^{\mathbb{P}}(x)) = \mathrm{tcl}(\mathrm{val}_{\mathcal{G}}(x))$.

Construction of Cohen terms for the stages of a progress.

Let e be a transitive set in the ground model of which $\mathbb{P}$ is a member, and let θ be indecomposable, exceeding the rank of e. P^e_θ is provident. Let $\dot{d}$ be the Cohen term $\hat{e} \cup \{\dot{\mathcal{G}}\}^{\mathbb{P}}$, so that $\mathrm{val}_{\mathcal{G}}(\dot{d})$ will be the transitive set $d = e \cup \{\mathcal{G}\}$.

REMARK $\dot{d}$ will be a member of $P^e_{\varrho(\mathbb{P})+k}$ for some (small) k, given the definition of $\dot{\mathcal{G}}$, our convention that $\mathbb{1} = 1$ and the fact that $\hat{\cdot}$ is $\mathbb{1}$-rud rec.

Our task is to build for each $\nu < \theta$ a name $N(\nu)$ for the stage P^d_ν of the progress towards d.

A simplified progress

Now $\varrho(\mathcal{G}) \leqslant \varrho(\mathbf{P}) < \varrho(\mathbb{P})$, so that for $\nu \geqslant \eta$, $d_\nu = e_\nu \cup \{\mathcal{G}\}$. It might be that $\varrho(\mathcal{G}) < \varrho(\mathbf{P})$; to avoid building names which make allowance for that uncertainty, we shall build names for the terms of a slightly different progress $(Q^d_\nu)_\nu$.

DEFINITION

$$\text{for } \nu < \eta, \quad Q^d_\nu = P^e_\nu; \qquad Q^d_\eta = P^e_\eta \cup \{\mathcal{G}\};$$

$$\text{for } \nu \geqslant \eta,\ Q^d_{\nu+1} = \mathbb{T}(Q^e_\nu) \cup \{d_\nu\} \cup d_{\nu+1};\ Q^d_\lambda = \bigcup_{\nu<\lambda} Q^d_\nu \text{ if } \lambda = \bigcup \lambda > \eta.$$

PROPOSITION *If θ is indecomposable, then Q^d_θ is provident and equals P^d_θ.*

Generic extensions of provident sets and of Jensen fragments

THEOREM *Let θ be an indecomposable ordinal strictly greater than the rank of a transitive set e which contains the notion of forcing, $\mathbb{P}$. Let $\mathcal{G}$ be $(P^e_\theta, \mathbb{P})$- generic. Then $(P^e_\theta)^{\mathbb{P}}[\mathcal{G}] = P^{e\cup\{\mathcal{G}\}}_\theta$ and hence is provident.*

Proof : $(P^e_\theta)^{\mathbb{P}}[\mathcal{G}]$ contains $P^{e\cup\{\mathcal{G}\}}_\theta$, as we have for each $\nu < \theta$ built a name in P^e_θ that evaluates under $\mathcal{G}$ to $Q^{e\cup\{\mathcal{G}\}}_\nu$, and we know by Proposition 8·4 that $Q^{e\cup\{\mathcal{G}\}}_\theta$ equals $P^{e\cup\{\mathcal{G}\}}_\theta$.

For the converse direction, we know that $P^{e\cup\{\mathcal{G}\}}_\theta$ is provident, and has $\mathcal{G}$ as a member and hence can support the $\mathcal{G}$-rudimentary recursion defining $\mathrm{val}_{\mathcal{G}}(\cdot)$. Further $P^{e\cup\{\mathcal{G}\}}_\theta$ includes $(P^e_\nu)_\nu$, which is defined by an e-rudimentary recursion, and so includes $(P^e_\theta)^{\mathbb{P}}[\mathcal{G}]$. $\dashv$

REMARK Thus, in this special case, a generic extension of a model of PROVI is a model of PROVI. We shall use this result to establish it more generally.

REMARK Theorem 9·0 remains true if the hypothesis on θ is weakened to requiring that $\theta > \varrho(\mathbb{P})$.

Proof that a generic extension of a provident set is provident.

THEOREM *Let A be provident, $\mathbb{P} \in A$ and $\mathcal{G}$ $(A, \mathbb{P})$-generic. Then $A^{\mathbb{P}}[\mathcal{G}]$ is provident.*

Proof : Let $\theta =_{\mathrm{df}} On \cap A$ and let

$$T = \{c \mid c \in A \;\&\; c \text{ is transitive } \;\&\; \mathbb{P} \in c\}.$$

Then

$$A = \bigcup \{P^c_\theta \mid c \in T\},$$

since the union on the right contains each element of A and is contained in A. It follows that

$$A^{\mathbb{P}}[\mathcal{G}] = \bigcup_{c \in T} (P^c_\theta)^{\mathbb{P}}[\mathcal{G}]$$

By Theorem 9·0, as each P^c_θ is provident and contains $\mathbb{P}$,

$$A^{\mathbb{P}}[\mathcal{G}] = \bigcup_{c\in T} P_\theta^{c\cup\{\mathcal{G}\}}$$

and each $P_\theta^{c\cup\{\mathcal{G}\}}$ is provident. Now in [M4, Proposition 5·52] we proved the

LEMMA *If θ is indecomposable and D is a collection of transitive sets each of rank less than θ and such that the pair of any two is a member of a third, then $\bigcup_{d\in D} P^d_\theta$ is provident.*

Take $D = \{c \cup \{\mathcal{G}\} \mid c \in T\}$ to complete the proof. $\dashv$

Forcing over provident models of Zermelo set theory: a commuting diagram

$$
\begin{array}{ccccc}
 & & H & \underset{\mathrm{val}_{\mathcal{G}}}{\longrightarrow} & H[\mathcal{G}] \\
 & & \uparrow \mathrm{Lune} & & \uparrow \mathrm{Lune} \\
\mathbb{P} & \in & I & \underset{\mathrm{val}_{\mathcal{G}}}{\longrightarrow} & I[\mathcal{G}]
\end{array}
$$

THEOREM *Let I and H be transitive sets such that I is a provident model of* Z *or at least* M, *and* $H = \mathrm{Lune}(I)$. *Let $\mathbb{P} \in I$ and let $\mathcal{G}$ be $(I, \mathbb{P})$ generic. Then $\mathcal{G}$ is $(H, \mathbb{P})$ generic, and further $H[\mathcal{G}] = \mathrm{Lune}(I[\mathcal{G}])$.*

Here the lune of a model of Zermelo or of Mac Lane set theory is a canonical extension of it which models the same theory plus Kripke-Platek.

R E F E R E N C E S

[De] K. Devlin, *Constructibility*, Perspectives in Mathematical Logic, *Springer-Verlag, Berlin*, 1984.

[G] R. O. Gandy, Set-theoretic functions for elementary syntax, in *Proceedings of Symposia in Pure Mathematics*, **13**, Part II, ed. T.Jech, American Mathematical Society, 1974, 103–126.

[H] K. Hauser, Generic Relativizations of Fine Structure, *Archive for Mathematical Logic* **39** (2000) 227–251.

[JK] R. B. Jensen and C. Karp, primitive recursive set functions, in *Proceedings of Symposia in Pure Mathematics*, **13**, Part I, ed. D. Scott, American Mathematical Society, 1971, 143–176.

[J1] R. B. Jensen, Stufen der konstruktiblen Hierarchie. Habilitationsschrift, Bonn, 1968.

[J2] R. B. Jensen, The fine structure of the constructible hierarchy, with a section by Jack Silver, *Annals of Mathematical Logic,* **4** (1972) 229–308; erratum ibid **4** (1972) 443.

[M1] A. R. D. Mathias, Slim models of Zermelo Set Theory, *Journal of Symbolic Logic* **66** (2001) 487–496.

[M2] A. R. D. Mathias, The Strength of Mac Lane Set Theory, *Annals of Pure and Applied Logic,* **110** (2001) 107–234.

[M3] A. R. D. Mathias, Weak systems of Gandy, Jensen and Devlin, in *Set Theory: Centre de Recerca Matemàtica, Barcelona 2003-4* , edited by Joan Bagaria and Stevo Todorcevic, Trends in Mathematics, Birkhäuser Verlag, Basel, 2006, 149–224.

[M4] A. R. D. Mathias, Rudimentary recursion and provident sets, *to be submitted*.

[M5] A. R. D. Mathias, Provident sets and rudimentary set forcing, *submitted*.

[M6] A. R. D. Mathias, Set forcing over models of Zermelo or Mac Lane, *submitted to the proceedings of the Brussels Centenary Conference on Zermelo Set Theory*.