Cofinal Types of Ultrafilters

Dilip Raghavan

University of Toronto

The Second European Set Theory Meeting, Będlewo
July 7, 2009

Outline

1. Tukey Reducibility
2. Basic Posets
3. Two Results

Basic Definitions

- Tries to capture the idea that two directed posets are “cofinally the same” (i.e. have the same cofinal type). Used for “rough classification” of directed posets

Definition

Given two directed posets $\mathbb{P}$ and $\mathbb{Q}$, we say that a map $f : \mathbb{P} \to \mathbb{Q}$ is a Tukey map if it maps (upward) unbounded sets in $\mathbb{P}$ to unbounded sets in $\mathbb{Q}$. We say that a map $g : \mathbb{Q} \to \mathbb{P}$ is a convergent map if the image of every (upward) cofinal subset of $\mathbb{Q}$ is cofinal in $\mathbb{P}$.

Basic Definitions

- Fact: There is a Tukey map $f : \mathbb{P} \to \mathbb{Q}$ iff there is a convergent $g : \mathbb{Q} \to \mathbb{P}$

Definition

We say $\mathbb{P}$ is Tukey reducible to $\mathbb{Q}$ and we write $\mathbb{P} \leq_T \mathbb{Q}$ if there is a a Tukey map $f : \mathbb{P} \to \mathbb{Q}$.

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- We have a natural equivalence $\mathbb{P} \equiv_T \mathbb{Q}$ iff $\mathbb{P} \leq_T \mathbb{Q}$ and $\mathbb{Q} \leq_T \mathbb{P}$. Then we say $\mathbb{P}$ and $\mathbb{Q}$ are *Tukey equivalent* and that $\mathbb{P}$ and $\mathbb{Q}$ have the same *Tukey type*.

Well known results

Possible Tukey types of directed sets of size $\aleph_1$ have been classified. There are both structure and non-structure results.

Theorem (Todorčević[2])

There are $2^{\mathfrak{c}}$ Tukey nonequivalent directed sets of size $\mathfrak{c}$. In particular, under CH, there are $2^{\aleph_1}$ cofinal types of size $\aleph_1$.

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Theorem (Todorčević[2])

Under PFA there are only 5 Tukey types of size $\aleph_1$: $1, \omega, \omega_1, \omega \times \omega_1, [\omega_1]^{<\omega}$

Ultrafilters

We want to investigate Tukey types of structures $\langle \mathcal{U}, \supset \rangle$, where $\mathcal{U}$ is an ultrafilter on ω. There are 3 motivations.

- First motivation: Tukey reducibility is a natural generalization of the well studied Rudin–Keisler reducibility.

Recall:

Definition

$\mathcal{V} \leq_{RK} \mathcal{U}$ if there is a function $f \in \omega^\omega$ such that for all $a \in [\omega]^\omega$, $a \in \mathcal{V}$ iff $f^{-1}(a) \in \mathcal{U}$.

Observation: If $f \in \omega^\omega$ witnesses that $\mathcal{V} \leq_{RK} \mathcal{U}$, the map $\phi : \mathcal{U} \to \mathcal{V}$ given by $\phi(a) = f''a$ is a convergent map (so $\mathcal{V} \leq_T \mathcal{U}$).

Ultrafilters

- Second motivation: Certain restricted classes of directed posets of size $\mathfrak{c}$, like Ultrafilters, may (consistently) have only few Tukey types. In fact, at the moment we do not know how to construct more than 2 Tukey types of ultrafilters in ZFC.

Ultrafilters

- Second motivation: Certain restricted classes of directed posets of size $\mathfrak{c}$, like Ultrafilters, may (consistently) have only few Tukey types. In fact, at the moment we do not know how to construct more than 2 Tukey types of ultrafilters in ZFC.
- Third motivation: Gives us an opportunity to apply the theory of basic posets invented by Solecki and Todorčević [1]. This is done in the recent paper of Dobrinen and Todorčević.

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- Third motivation: Gives us an opportunity to apply the theory of basic posets invented by Solecki and Todorčević [1]. This is done in the recent paper of Dobrinen and Todorčević.
- The two Tukey types known to exist in ZFC are 1 (principal ultrafilters) and $[\mathfrak{c}]^{<\omega}$.
- $\mathcal{U} \equiv_T [\mathfrak{c}]^{<\omega}$ iff there is a set $\{x_\alpha : \alpha < \mathfrak{c}\} \subset \mathcal{U}$ such that for every $A \in [\mathfrak{c}]^{\omega}$, $\bigcap_{\alpha \in A} x_\alpha \notin \mathcal{U}$.

Ultrafilters

- Fact(Isbell): There is (in ZFC) a $\mathcal{U} \equiv_T [\mathfrak{c}]^{<\omega}$. Let $\{x_\alpha : \alpha < \mathfrak{c}\}$ be an independent family, and take $\mathcal{U}$ which extends $\{x_\alpha : \alpha < \mathfrak{c}\} \cup \{\bigcup_{\alpha \in A}(\omega \setminus x_\alpha) : A \in [\mathfrak{c}]^{\omega}\}$
- Open Question(Isbell): Can one construct in ZFC a $\mathcal{U} <_T [\mathfrak{c}]^{<\omega}$?

Basic Posets

Definition

Let $\langle \mathbb{P}, \leq \rangle$ be a directed poset with a separable metric topology. $\mathbb{P}$ is said to be basic if

1. *Every convergent sequence in $\mathbb{P}$ has an infinite subsequence that is bounded in $\mathbb{P}$*
2. *Every bounded sequence in $\mathbb{P}$ has an infinite subsequence that converges in $\mathbb{P}$.*

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Theorem (Solecki and Todorčević [1])

Let $\mathbb{P}$ and $\mathbb{Q}$ be basic posets and suppose $\mathbb{P} \leq_T \mathbb{Q}$. Then there is a Borel monotone $\phi : \mathbb{Q} \to \mathbb{P}$ such that $\phi''\mathbb{Q}$ is cofinal in $\mathbb{P}$.

Basic Posets

Definition

A monotone $\phi : \mathbb{Q} \to \mathbb{P}$ *such that* $\phi''\mathbb{Q}$ *is cofinal in* $\mathbb{P}$ *is called a cofinal map. Every cofinal map is convergent.*

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So which ultrafilters are basic (with the usual topology)? Property 1 is the main point.
Fact: $\mathcal{U}$ is basic iff $\mathcal{U}$ is a P-point.
So if $\mathcal{U}$ and $\mathcal{V}$ are P-points and $\mathcal{V} \leq_T \mathcal{U}$, then there is a Borel cofinal $\phi : \mathcal{U} \to \mathcal{V}$.

P-points

Theorem

Suppose $\mathcal{U}$ is a P-point and $\mathcal{V}$ is an ultrafilter such that $\mathcal{V} \leq_T \mathcal{U}$. Then there is a set $a \in \mathcal{U}$ and a continuous cofinal map $\phi : \mathcal{U} \cap a \to \mathcal{V}$.

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First of all, suppose $\psi : \mathcal{V} \to \mathcal{U}$ is a Tukey map. For $a \in \mathcal{U}$, put $X(a) = \{e \in \mathcal{V} : \psi(e) \supset a\}$. Notice that $X(a)$ is bounded in $\mathcal{V}$ because $\psi'' X(a)$ is bounded in $\mathcal{U}$ by a.
So $\bigcap X(a) \in \mathcal{V}$. Define $\phi : \mathcal{U} \to \mathcal{V}$ by $\phi(a) = \bigcap X(a)$.

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So $\bigcap X(a) \in \mathcal{V}$. Define $\phi : \mathcal{U} \to \mathcal{V}$ by $\phi(a) = \bigcap X(a)$.
Now, ϕ is monotone, because if $a \subset b$, then $X(b) \subset X(a)$.
$\phi'' \mathcal{U}$ is cofinal in $\mathcal{V}$ because $e \in X(\psi(e))$ for all $e \in \mathcal{V}$.

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Lemma

Suppose $\mathcal{U}$ is a P-point and $\mathcal{V}$ is an ultrafilter. Let $\phi : \mathcal{U} \to \mathcal{V}$ is a cofinal map. Then there is a set $a \in \mathcal{U}$ such that $\phi \restriction \mathcal{U} \cap a$ is continuous.

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Recall the P-point game: $a_0, s_0, a_1, s_1, \cdots a_{n-1}, s_{n-1}$
Player I chooses a set $a_n \in \mathcal{U}$ (where $\mathcal{U}$ is an arbitrary ultrafilter) and Player II responds with a finite subset $s_n \subset a_n$.
I wins iff $\bigcup s_n \notin \mathcal{U}$.

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Fact

$\mathcal{U}$ is a P-point iff I does not have a winning strategy in the P-point game.

P-points

Given a monotone $\phi : \mathcal{U} \to \mathcal{V}$, we define a strategy for I as follows. Given $b_0, s_0, b_1, s_1, \cdots b_{n-1}, s_{n-1}$, put $s = s_0 \cup \cdots \cup s_{n-1}$. Player I chooses a $b_n \subset b_{n-1}/s$, with $b_n \in \mathcal{U}$ such that for every $t \subset s$ *either* $\forall c \in \mathcal{U} \cap b_n \, [n \in \phi(t \cup c)]$ *or* $\forall c \in \mathcal{U} \cap b_n \, [n \notin \phi(t \cup c)]$. This is possible by monotonicity of ϕ. Given $t \subset s$, Player I just asks "Is there a $c \in \mathcal{U}$ such that $n \notin \phi(t \cup c)$"? If the answer is yes, I chooses such a $c_t \in \mathcal{U}$. Note any subset of c_t will have this property too. I plays $b_n = \left(\bigcap c_t\right) \cap \left(b_{n-1}/s\right)$.

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If $\mathcal{U}$ is a P-point, then I has no winning strategy. So there is a play $a_0, s_0, a_1, s_1, \cdots$ according to the above strategy that I looses. So $a = \bigcup s_n \in \mathcal{U}$ and $\phi \restriction \mathcal{U} \cap a$ is continuous because for any $c \in \mathcal{U} \cap a$ whether $n \in \phi(c)$ or not is determined by $c \cap \bigcup_{m<n} s_m$.

P-points

- So there are only $\mathfrak{c}$ cofinal maps to consider (even though there are $2^{\mathfrak{c}}$ P-points).
- This can be used along with CH (or MA) to embed various structures into the class of P-points with the $<_T$ relation because all possible cofinal maps can be enumerated in type ω_1.

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- So there are only $\mathfrak{c}$ cofinal maps to consider (even though there are $2^{\mathfrak{c}}$ P-points).
- This can be used along with CH (or MA) to embed various structures into the class of P-points with the $<_T$ relation because all possible cofinal maps can be enumerated in type ω_1.
- This has partially been done in a recent paper of Dobrinen and Todorčević.
- For example one of their results is that using CH one can embed ω_1 into the P-points under $<_T$ – i.e. there is a strictly increasing chain of P-points $\mathcal{U}_0 <_T \mathcal{U}_1 <_T \cdots <_T \mathcal{U}_\alpha <_T \cdots$ for $\alpha < \omega_1$

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Is Tukey theory different from Rudin-Keisler theory on the P-points?

Motivation: We know that if $\mathcal{U}$ is a P-point and $\mathcal{V} \leq_T \mathcal{U}$, then there is a "nice" cofinal map $\phi : \mathcal{U} \to \mathcal{V}$. Can we make this map even nicer and turn it into a Rudin-Keisler map?

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If $\mathcal{U}$ is a selective ultrafilter, then is easy to see that $\mathcal{U} \times \mathcal{U} \equiv_T \mathcal{U}$, but we can never have $\mathcal{U} \times \mathcal{U} \leq_{RK} \mathcal{U}$. So we restrict to the class of P-points only.

Two Results

Theorem

Assuming CH *there is a sequence of P-points* $\langle \mathcal{U}_r : r \in \mathbb{R} \rangle$ *such that whenever* $r < s$, $\mathcal{U}_r <_T \mathcal{U}_s$.

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Theorem

Assume CH. *There exist P-points* $\mathcal{U}$ *and* $\mathcal{V}$ *such that* $\mathcal{V} <_{RK} \mathcal{U}$, *but* $\mathcal{V} \equiv_T \mathcal{U}$.

Two Results

Build $\mathcal{U}$ on $\omega \times \omega$. $\mathcal{V}$ is the projection of $\mathcal{U}$ onto the first coordinate. If we make sure that $\mathcal{U}$ is a P-point, then $\mathcal{V}$ is also be a P-point and $\mathcal{V} \leq_{RK} \mathcal{U}$.

Definition

Let $E \subset \omega \times \omega$. We define $E(n) = \{m \in \omega : \langle n, m \rangle \in E\}$. Also define

$$\mathcal{E}_0 = \{E \subset \omega \times \omega : \forall k \in \omega \exists^\infty n \in \omega \left[\left| E(n) \right| > k \right] \}.$$

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Fact: If $\mathcal{U}$ is a P-point with $\mathcal{U} \subset \mathcal{E}_0$, and if $\mathcal{V} = \pi_0(\mathcal{U})$, then $\mathcal{U} \not\leq_{RK} \mathcal{V}$. So to ensure that $\mathcal{V} <_{RK} \mathcal{U}$, it is enough to build $U \subset \mathcal{E}_0$.

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Fact: If $\mathcal{U}$ is a P-point with $\mathcal{U} \subset \mathcal{E}_0$, and if $\mathcal{V} = \pi_0(\mathcal{U})$, then $\mathcal{U} \not\leq_{RK} \mathcal{V}$. So to ensure that $\mathcal{V} <_{RK} \mathcal{U}$, it is enough to build $U \subset \mathcal{E}_0$.

But what about $U \equiv_T \mathcal{V}$? We must have $\mathcal{U} \leq_T \mathcal{V}$. So we must fix a monotone continuous map $\phi : [\omega]^\omega \to [\omega \times \omega]^\omega$ *in advance* and build $\mathcal{U}$ so that $\phi'' \pi_0(\mathcal{U})$ is cofinal in $\mathcal{U}$.

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$$\forall \langle n, m \rangle \in \omega \times \omega \left[\langle n, m \rangle \in \phi_f(a) \iff (n, m \in a) \wedge (m < n) \wedge (f(m) = f(n))\right].$$

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Question: Is Tukey theory the same as RK theory within the class of selective ultrafilters?

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