Equivalence relations generated by profinite actions

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Université Pierre et Marie Curie – Paris 6

Second European Set Theory Meeting, Będlewo 2009

Definable equivalence relations

Basic setting:

- X – standard Borel space (the interval $[0,1]$ equipped with its Borel σ-algebra);
- E – "definable" (Borel or analytic) equivalence relation on X.

One is interested in the quotient space $X/E = \{[x]_E : x \in X\}$ and, more concretely, in the maps between different quotient spaces.

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Examples:

- $(\mathbf{R}, =)$;
- $(2^{\mathbf{N}}, E_0)$ – two elements of $2^{\mathbf{N}}$ are E_0 equivalent if their symmetric difference is finite;
- orbit equivalence relations: G – Polish group, $G \curvearrowright X$ is a Borel action, the orbit equivalence relation is analytic;
- Turing equivalence on $2^{\mathbf{N}}$.

Homomorphisms

A Borel map $f: X \to Y$ is called a **homomorphism** from E to F if

$$\forall x_1, x_2 \in X \quad x_1 \, E \, x_2 \implies f(x_1) \, F \, f(x_2).$$

A homomorphism f is a **reduction** if

$$\forall x_1, x_2 \in X \quad x_1 \, E \, x_2 \iff f(x_1) \, F \, f(x_2),$$

i.e., f is injective if considered as a map between the quotient spaces.

E is called **Borel reducible** to F (denoted by $E \leq_B F$) if such a reduction exists.

A large part of the theory is concerned with determining whether such reductions exist between some concrete equivalence relations.

Countable Borel equivalence relations and ergodic theory

- An equivalence relation is called **countable** if all of its equivalence classes are countable.
- By a standard result of Feldman and Moore (an easy consequence of the Lusin–Novikov uniformization theorem), every countable Borel equivalence relation is the orbit equivalence relation of a Borel action of a countable group.

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- If, in addition, one equips the space X with a measure invariant with respect to the equivalence relation, one can use powerful tools from ergodic theory and, most notably, cocycle superrigidity theorems to prove non-reducibility results.
- However, if one uses the measure, one loses all information on null sets, so it is impossible to prove positive results with those techniques.

Measures, ergodicity, hyperfiniteness

A **standard probability space** is a standard Borel space equipped with a Borel, non-atomic probability measure (for example $([0,1],\lambda)$).

Let E be a countable Borel equivalence relation on a standard probability space (X,μ).

- E is **measure-preserving** if for every partial bijection $\tau \subseteq E$, $\mu(\operatorname{dom}\tau) = \mu(\operatorname{rng}\tau)$.
- E is **ergodic** if every measurable invariant set (union of equivalence classes) is null or co-null.
- Ornstein–Weiss: If E is the orbit equivalence relation of a group action $\Gamma \curvearrowright (X,\mu)$, where the group Γ is *amenable*, then E is hyperfinite μ-a.e.

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- It is not known whether in the above situation E must be hyperfinite *everywhere*.

Orbit equivalence

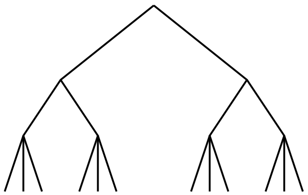
Definition

Two measure-preserving equivalence relation E and F on spaces (X, μ) and (Y, ν), respectively are **isomorphic** if there are invariant Borel sets $A \subseteq X$ and $B \subseteq Y$ of full measure and a measure-preserving map $f : A \to B$ such that

$$x_1 \, E \, x_2 \iff f(x_1) \, F \, f(x_2).$$

Orbit equivalence has become an important meeting point of ergodic theory, Borel equivalence relations, and von Neumann algebras.

Profinite group actions



Let $\mathcal{T}$ be a finitely splitting rooted tree and consider an action $\Gamma \curvearrowright \mathcal{T}$ by automorphisms. If $X = [\mathcal{T}]$, then X can be equipped with a natural compact topology, a Borel probability measure, and a measure-preserving action of Γ. The resulting measure-preserving action $\Gamma \curvearrowright X$ is called **profinite**.

A profinite action is ergodic iff the action on the tree is transitive on every level.

Say that an equivalence relation is **profinite** if it is isomorphic to an equivalence relation generated by a profinite action.

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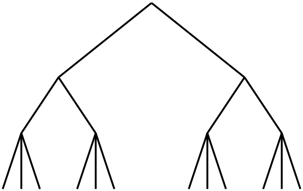

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Example

Dye's theorem implies that all hyperfinite equivalence relations are profinite.

Examples of profinite actions

A group Γ is **residually finite** if there exists a sequence of subgroups

$$\Gamma \geq \Gamma_1 \geq \Gamma_2 \geq \cdots$$

of finite index such that $\bigcap_n \Gamma_n = \{1\}$.

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Every residually finite group has profinite actions: consider the *coset tree* corresponding to the sequence $\Gamma_1, \Gamma_2, \ldots$: level n of the tree consists of the left cosets Γ/Γ_n and the tree structure is given by inclusion.

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Example

Consider the sequence of subgroups

$$\mathbf{Z} \geq 2\mathbf{Z} \geq 2^2\mathbf{Z} \geq \cdots.$$

The corresponding profinite action is nothing but the action $\mathbf{Z} \curvearrowright \mathbf{Z}_2$ by translation. ($\mathbf{Z}_2 = \varprojlim \mathbf{Z}/2^n\mathbf{Z}$)

Non-profinite equivalence relations?

A homomorphism between two equivalence relations is **countable-to-one** (also called a **weak reduction**) if the preimage of every point is countable. Every reduction between countable equivalence relations is countable-to-one.

Theorem (Hjorth)

Let E be the orbit equivalence relation of the Bernoulli shift $\mathbf{F}_2 \curvearrowright 2^{\mathbf{F}_2}$ and let F be an arbitrary profinite equivalence relation. Let $A \subseteq 2^{\mathbf{F}_2}$ be a conull set (with respect to the Bernoulli measure). Then there does not exist a countable-to-one homomorphism from $E|_A$ to F.

The theorem also holds for groups Γ containing $\mathbf{F}_2$.

The Koopman representation

Let $\Gamma \curvearrowright^a (X, \mu)$ be a measure-preserving group action. It gives rise to a natural unitary representation $\Gamma \curvearrowright L^2(X, \mu)$, called the **Koopman representation** of the action.

It always has a trivial invariant one-dimensional subspace (the constants), so one usually considers the restriction of the representation to its orthogonal complement and denotes it by $\kappa_0(a)$.

The Koopman representation captures many features of the action and is a classical object of study in ergodic theory. Properties that can be expressed using the Koopman representation are called **spectral properties** of the action (those include ergodicity, weak mixing, mixing, compactness, etc.)

Tempered actions

Theorem (Kechris)

Suppose that $\mathbf{F}_2 \leq \Gamma$ and $\Gamma \curvearrowright (X, \mu)$ is a measure-preserving group action such that $\kappa_0 \prec \lambda_\Gamma$ and denote by E the orbit equivalence relation. Then for any conull set $A \subseteq X$, there does not exist a countable-to-one homomorphism from $E|_A$ to an equivalence relation generated by a profinite action.

Question (Kechris)

Can the condition of containing a free group be replaced by non-amenability?

Amenable unitary representations

Let Γ be a countable group and $\pi\colon \Gamma \to U(\mathcal{H})$ be a unitary representation of Γ on a Hilbert space $\mathcal{H}$.

π is said to be **amenable** if there exists a $\pi(\Gamma)$-invariant mean on $B(\mathcal{H})$, i.e., a positive linear functional $M\colon B(\mathcal{H}) \to \mathbf{C}$ such that

- $M(I) = 1$ and
- $M(\pi(\gamma)T\pi(\gamma)^{-1}) = M(T)$ for all $T \in B(\mathcal{H}), \gamma \in \Gamma$.

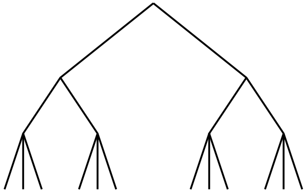
Equivalently, π is amenable iff $1 \prec \pi \otimes \overline{\pi}$ (Bekka).

Every finite dimensional representation is amenable and

$$\pi \leq \rho \text{ and } \pi \text{ is amenable} \implies \rho \text{ is amenable.}$$

The Koopman representation of a profinite action

Consider a profinite action $\Gamma \curvearrowright (X, \mu)$.



Let $\mathcal{A}_n$ be the partition of X corresponding to the n-th level of the tree and consider the finite dimensional Hilbert space

$$\mathcal{H}_n = \{f \in L^2(X) : f \text{ is constant on each } A \in \mathcal{A}_n\}.$$

There is a natural representation π_n of Γ on $\mathcal{H}_n$ and

$$\kappa_0 = \varinjlim \pi_n.$$

A non-amenability obstruction for profiniteness

Theorem (Epstein–Ts. (2007))

Let $\Gamma \curvearrowright (X, \mu)$ be a measure preserving action such that its Koopman representation is non-amenable. Then if E denotes its orbit equivalence relation and $A \subseteq X$ is a conull set, there does not exist a countable-to-one homomorphism from $E|_A$ to a profinite equivalence relation.

Examples of actions for which the theorem holds:

- Bernoulli shifts of non-amenable groups ($\Gamma \curvearrowright 2^\Gamma$);
- mixing actions of groups without the Haagerup approximation property;
- weakly mixing actions of groups with property (T).

Almost invariant partitions I

Definition

Say that a group action $\Gamma \curvearrowright (X, \mu)$ admits **almost invariant partitions** if for every partition $\mathcal{A}$ of X, every finite set $Q \subseteq \Gamma$ and every $\epsilon > 0$, there exists a partition $\mathcal{B}$ of X such that:

- $\mathcal{B}$ almost refines $\mathcal{A}$: for all $A \in \mathcal{A}$, there exists $B \subseteq X$ which is a union of elements of $\mathcal{B}$ such that $\mu(A \triangle B) < \epsilon$;
- $\mathcal{B}$ is (Q, ϵ)-almost invariant: for all $T \in Q$, there exists a permutation $\sigma \in S(\mathcal{B})$ such that

$$\sum_{B \in \mathcal{B}} \mu(T(B) \triangle \sigma B) < \epsilon.$$

Profinite actions obviously admit almost invariant partitions.

Almost invariant partitions II

Proposition

Admitting almost invariant partitions is an invariant of orbit equivalence.

Definition

An equivalence relation admits **almost invariant partitions** if some (or, equivalently, any) group action $\Gamma \curvearrowright X$ that generates it does.

A characterization

Theorem

Let E be an ergodic equivalence relation on (X, μ). Then the following are equivalent:

- E admits almost invariant partitions;
- E is profinite.

Corollaries:

- If an equivalence relation admits a countable-to-one homomorphism into a profinite equivalence relation, then it is profinite.
- In particular, subequivalence relations and complete sections of profinite equivalence relations are themselves profinite.

Sofic groups

- A group Γ is **residually finite** if homomorphisms from Γ into finite groups separate points: for every finite set $Q \subseteq \Gamma$, there exists $n \in \mathbf{N}$ and a homomorphism $\phi \colon \Gamma \to S_n$ such that for all $\gamma \in Q \setminus \{1\}$, $\phi(\gamma) \neq 1$.
- A group is called **sofic** if almost homomorphisms into finite groups separate points. More precisely, G is **sofic** if for all finite $Q \subseteq G$ and all $\epsilon > 0$, there exists $n \in \mathbf{N}$ and a map $\phi \colon Q \to S_n$ such that
  - for all $g \in Q \setminus \{1\}$, $\quad d(\phi(g), 1) > 1/2$;
  - for all $g, h, gh \in Q$, $\quad d(\phi(gh), \phi(g)\phi(h)) < \epsilon$,

  where $d(\sigma, \tau) = (1/n)|\{\sigma \neq \tau\}|$ denotes the normalized Hamming distance on S_n.

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- A group is sofic iff every finitely generated subgroup of it is.
- It is not known whether non-sofic groups exist.

Profinite actions and sofic groups

Definition

Let E be a measure-preserving equivalence relation on (X, μ). The **full group** of E is defined by

$$[E] = \{T \in \mathrm{Aut}(X, \mu) : \mathrm{graph}\, T \subseteq E\}.$$

Proposition

If E admits almost invariant partitions, then $[E]$ is sofic (equivalently, every countable subgroup of $[E]$ is sofic).

Corollary

If E admits a countable-to-one homomorphism into a profinite equivalence relation, then E cannot be generated by *any* action of a non-sofic group.

Compact actions

A measure-preserving group action $\Gamma \curvearrowright (X, \mu)$ is called **compact** if its Koopman representation is a sum of finite-dimensional representations. For example, profinite actions are compact.

In fact, every compact, ergodic action is isomorphic to an action of the form $\Gamma \curvearrowright K/L$, where K is a compact group, L is a closed subgroup of K, K/L is equipped with the quotient of the Haar measure on K, Γ is a dense subgroup of K, and the action is by translation.

Question: Are all orbit equivalence relations of compact actions profinite?