The search for the ultimate enlargement of L

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The Inner Model Program

The Inner Model Problem

Find generalizations of Gödel's L in which large cardinal axioms hold.

Theorem (Scott)

Suppose that there is a measurable cardinal. Then $V \neq L$.

Corollary

$V \neq L$.

The building blocks for inner models: Extenders

Suppose that

$$j : V \to M$$

is an elementary embedding with critical point κ, $\kappa < \gamma \leq j(\kappa)$, and that

$$V_\gamma \subset M.$$

Extracting the measure μ_a induced by j

For each finite set $a \subset \gamma$ let μ_a be the ultrafilter on $[\kappa]^n$ such that $\mu_a(A) = 1$ if and only if $a \in j(A)$ where $n = |a|$.

The extender E of length γ derived from j

The extender *E of length γ defined from j is the sequence*

$$\langle E_a : a \in [\gamma]^{<\omega} \rangle.$$

where $E_a = \mu_a$ for each $a \in [\gamma]^{<\omega}$

The ultrapower $\mathrm{Ult}(V, E)$

For each $a \in \mathrm{dom}(E)$ one can define $\mathrm{Ult}(V, E_a)$ and then take a direct limit under the natural maps to obtain an elementary embedding

$$j_E : V \to M_E.$$

There exists a factor elementary embedding

$$k_E : M_E \to M$$

such that $\mathrm{CRT}(k_E) \geq \gamma$ and $j = k_E \circ j_E$.

Martin-Steel Extender Sequences

An extender sequence,

$$\tilde{E} = \langle E^\alpha_\beta : (\alpha, \beta) \in \mathrm{dom}(\tilde{E}) \rangle$$

is a *Martin-Steel extender sequence* if for each pair $(\alpha, \beta) \in \mathrm{dom}(\tilde{E})$ the following hold

Coherence Condition

There exists an extender F such that

1. $\alpha < \mathrm{LTH}(F)$,
2. $E^\alpha_\beta = F|\alpha$
3. $j_F(\tilde{E})|(\alpha + 1, 0) = \tilde{E}|(\alpha, \beta)$.

Novelty Condition

For all $\beta^* < \beta$, $(\alpha, \beta^*) \in \mathrm{dom}(\tilde{E})$ and

$$E^\alpha_{\beta^*} \cap L[\tilde{E}|(\alpha, \beta)] \neq E^\alpha_\beta \cap L[\tilde{E}|(\alpha, \beta)]$$

Initial Segment Condition

Suppose that

$$\kappa < \alpha^* < \alpha$$

where κ *is the critical point associated to* E^{α}_{β}.
Then there exists β^* *such that* $(\alpha^*, \beta^*) \in \mathrm{dom}(\tilde{E})$ *and such that*

$$E^{\alpha^*}_{\beta^*} \cap L[\tilde{E}|(\alpha^* + 1, 0)] = (E^{\alpha}_{\beta}|\alpha^*) \cap L[\tilde{E}|(\alpha^* + 1, 0)].$$

Lemma (Martin, Steel)

Suppose that there exists an elementary embedding

$$j : V \to M$$

such that $V_{j(\kappa)+1} \subset M$ *where* $\kappa = \mathrm{CRT}(j)$. *Then there exists a Martin-Steel extender sequence* $\tilde{E}$ *such that* κ *is a superstrong cardinal in* $L[\tilde{E}]$.

Definition

Suppose $\tilde{E}$ is a Martin-Steel extender sequence.

- $T_{\tilde{E}}$ is the Σ_1-theory of the structure

$$\left(L[\tilde{E}], \tilde{E} \cap L[\tilde{E}]\right).$$

Definition

Suppose $\tilde{E}$ and $\tilde{F}$ are Martin-Steel extender sequences.

- *Comparison* holds for $(\tilde{E}, \tilde{F})$ if either $T_{\tilde{E}} \subseteq T_{\tilde{F}}$ or $T_{\tilde{F}} \subseteq T_{\tilde{E}}$.

Definition

Suppose $\tilde{E}$ is a Martin-Steel extender sequence.

- $\tilde{E}$ is 1-*small* if for all $(\alpha, \beta) \in \mathrm{dom}(\tilde{E})$,

$$L[\tilde{E}|(\alpha, \beta)] \not\models \text{“There is a Woodin cardinal”}$$

Theorem (Martin, Steel)

Suppose there is a Woodin cardinal.

- *Then there is a 1-small Martin-Steel extender sequence such that*

$$L[\tilde{E}] \models \text{“There is a Woodin cardinal”}$$

Theorem (Martin, Steel)

Assume there is a proper class of measurable cardinals.

- *Comparison holds for all pairs of 1-small Martin-Steel extender sequences.*

Iteration Hypothesis (IH)

Suppose $X \prec V_\Theta \prec_{\Sigma_2} V$ *and* X *is countable. Let* M *be the transitive collapse of* X*. Then*

- M *is* $(\omega_1 + 1)$*-iterable.*

- The notion of iteration involves *iteration trees*.

Theorem (Martin, Steel)

Assume IH*. Then comparison holds for all pairs of Martin-Steel extender sequences.*

Suppose $\tilde{E}$ is a Martin-Steel extender sequence of length 1. Then

$$L[\tilde{E}] = L[E] = L[\mu]$$

and so $L[\tilde{E}]$ is uniquely determined by $\mathrm{dom}(\tilde{E})$ since in this case

$$\mathrm{dom}(\tilde{E}) = \{(\alpha, 0)\}$$

where $\alpha = \mathrm{CRT}(E) + 1$.

Theorem

Assume IH. *Suppose that $\tilde{E}$ and $\tilde{F}$ are Martin-Steel extender sequences such that $\mathrm{dom}(\tilde{E}) = \mathrm{dom}(\tilde{F})$.*

- *Then $L[\tilde{E}] = L[\tilde{F}]$ and for all $(\alpha, \beta) \in \mathrm{dom}(\tilde{E})$,*

$$E^{\alpha}_{\beta} \cap L[\tilde{E}] = F^{\alpha}_{\beta} \cap L[\tilde{F}].$$

Definition (Solovay)

A cardinal δ is *supercompact* if for all α there exists an elementary embedding,

$$j : V \to M$$

such that $\mathrm{CRT}(j) = \delta$, $j(\delta) > \alpha$, and such that $M^\alpha \subset M$.

Definition

Suppose $\tilde{E}$ is a Martin-Steel extender sequence. Then $\tilde{E}^*$ is the maximal Martin-Steel extender sequence one can construct in $L[\tilde{E}]$ using extenders from $\tilde{E}$ to witness the Coherence Condition.

Theorem

Suppose $\tilde{E}$ is a Martin-Steel extender sequence such that

(1) $\mathbb{R} \cap L[\tilde{E}] = \mathbb{R} \cap L[\tilde{E}^*]$,

(2) $L[\tilde{E}] \models \mathrm{IH}$.

Then $L[\tilde{E}] \not\models$ "There is a supercompact cardinal".

Long extenders

Suppose that

$$j : V \to M$$

is an elementary embedding with critical point κ, $\kappa < \gamma$, and that

$$V_\gamma \subset M.$$

Let $\hat{\gamma}$ be the least ordinal α such that $\gamma \leq j(\alpha)$.

Extracting the measure μ_a induced by j

For each finite set $a \subset \gamma$ let E_a be the ultrafilter on $[\hat{\gamma}]^n$ such that $E_a(A) = 1$ if and only if $a \in j(A)$ where $n = |a|$.

The (long) extender E of length γ derived from j

The extender *E of length γ defined from j is the sequence*

$$\langle E_a : a \in [\gamma]^{<\omega} \rangle.$$

Generalized Martin-Steel extender sequences

Definition

An extender sequence $\tilde{E}$ is a *generalized Martin-Steel extender sequence* if it satisifies the Coherence, Novelty, and Initial Segment conditions as in the definition of a Martin-Steel extender sequence.

Theorem (Martin, Steel)

Suppose that for each $n < \omega$ there is a proper class of n-huge cardinals.

- *Then for each $n < \omega$ there is a generalized Martin-Steel extender sequence $\tilde{E}$ such that*

$$L[\tilde{E}] \models \text{“}\textit{There is an n-huge cardinal}\text{”}.$$

Definition

Suppose $\tilde{E}$ and $\tilde{F}$ are generalized Martin-Steel extender sequences.

- $T_{\tilde{E}}$ is the Σ_1-theory of the structure $\left(L[\tilde{E}], \tilde{E} \cap L[\tilde{E}]\right)$.
- *Comparison* holds for $(\tilde{E}, \tilde{F})$ if either $T_{\tilde{E}} \subseteq T_{\tilde{F}}$ or $T_{\tilde{F}} \subseteq T_{\tilde{E}}$.

Definition

A **critical pair of extenders** is a pair (E, F) such that

$$j_E(\mathrm{CRT}(E)) < \mathrm{CRT}(F) \leq \mathrm{LTH}(E).$$

Steel's moving spaces problem

The basic methodology for establishing comparision fails for generalized Martin-Steel extender sequences if the sequences contain critical pairs.

Definition

A generalized Martin-Steel extender sequence is **strongly backgrounded** if

- for each $(\alpha, \beta) \in \mathrm{dom}(\tilde{E})$ and for each γ, there is an extender F such that $\mathrm{LTH}(F) > \gamma$ and such that F witnesses the Coherence Condition for E^{α}_{β}.

Definition

1. $\mathcal{E}^{(\infty)}$ is the set of all generalized Martin-Steel sequences $\tilde{E}$ such that $\tilde{E}$ contains a critical pair of extenders and such that no proper intial segment of $\tilde{E}$ conains a critical pair.
2. $\mathcal{E}^{+}_{(\infty)}$ denotes the set of $\tilde{E} \in \mathcal{E}^{(\infty)}$ such that $\tilde{E}$ is strongly backgrounded.

Definition

A cardinal δ is an extendible cardinal if for each α there exists an elementary embedding

$$j : V_{\delta+\alpha+1} \to V_{j(\delta)+j(\alpha)+1}$$

such that $\mathrm{CRT}(j) = \delta$ and $j(\delta) > \alpha$.

Theorem

Suppose there is an extendible cardinal. Then

$$\mathcal{E}^{+}_{(\infty)} \neq \emptyset.$$

The failure of comparison

Theorem

Suppose that there is a proper class of huge cardinals, $\gamma \in \mathrm{Ord}$ *and that* γ *is* Σ_2*-definable.*

- *There exists a transitive set* M *such that*
  (1) $M \models \mathrm{ZFC}$ *and* $V_\gamma \in M$,
  (2) $M \models$ "*There is an extendible cardinal*",
  (3) *for all* $\tilde{E} \in \mathcal{E}^+_{(\infty)}$, *for all* $\tilde{F} \in \left(\mathcal{E}^+_{(\infty)}\right)^M$, $T_{\tilde{E}} \not\subseteq T_{\tilde{F}}$ *and* $T_{\tilde{F}} \not\subseteq T_{\tilde{E}}$.

- This theorem effectively rules out inner model theory for extender models at the level of a critical pair.

Suitable Extender Sequences

Definition

A suitable extender sequence is [... a generalized Martin-Steel extender sequence which contains no critical pairs ...]

- The actual definition of a suitable extender sequence is slightly different; based on a reorganization of the sequence.

Definition

Suppose $\tilde{E}$ is a suitable extender sequence of class length. Then

$$o^{\tilde{E}}_{\mathrm{LONG}}(\delta) = \infty$$

if for all $\alpha > \delta$ there exists $(\alpha, \beta) \in \mathrm{dom}(\tilde{E})$ such that

$$j_E(\mathrm{CRT}(E)) = \delta$$

where $E = E^{\alpha}_{\beta}$.

Lemma

Suppose $\tilde{E}$ is a suitable extender sequence such that

$$o^{\tilde{E}}_{\mathrm{LONG}}(\delta) = \infty.$$

(1) *Suppose $(\alpha, \beta) \in \mathrm{dom}(\tilde{E})$. Then $\mathrm{CRT}(E^{\alpha}_{\beta}) < \delta$.*

(2) *Suppose $o^{\tilde{E}}_{\mathrm{LONG}}(\kappa) = \infty$ then $\delta = \kappa$.*

Lemma

Suppose $\tilde{E}$ is a suitable extender sequence such that

$$o^{\tilde{E}}_{\mathrm{LONG}}(\delta) = \infty.$$

- *Then $L[\tilde{E}] \models$ "δ is supercompact".*

Lemma (Weak Covering)

Suppose $\tilde{E}$ is a suitable extender sequence such that

$$o^{\tilde{E}}_{\mathrm{LONG}}(\delta) = \infty.$$

Suppose $\gamma > \delta$ and γ is a singular cardinal.

- *Then γ is singular in $L[\tilde{E}]$ and $\gamma^+ = (\gamma^+)^{L[\tilde{E}]}$.*

So the weak version Jensen's Covering Lemma **must** *hold for $L[\tilde{E}]$ with* **no limiting** *assumptions on V.*

A remarkable coincidence

Lemma (Closure under extenders)

Suppose $\tilde{E}$ *is a suitable extender sequence such that* $o^{\tilde{E}}_{\mathrm{LONG}}(\delta) = \infty$. *Suppose* $\gamma > \delta$ *and*

$$j : L[\tilde{E}] \cap V_{\gamma+1} \to L[\tilde{E}] \cap V_{j(\gamma)+1}$$

is an elementary embedding with $\mathrm{CRT}(j) \geq \delta$. *Then* $j \in L[\tilde{E}]$.

Corollary (Rigidity)

Suppose $\tilde{E}$ *is a suitable extender sequence such that* $o^{\tilde{E}}_{\mathrm{LONG}}(\delta) = \infty$.

- *Then there is no non-trivial elementary embedding*

  $$j : L[\tilde{E}] \to L[\tilde{E}]$$

  with $\mathrm{CRT}(j) > \delta$.

Lemma (Generic closure)

Suppose $\tilde{E}$ is a suitable extender sequence such that

$$o^{\tilde{E}}_{\mathrm{LONG}}(\delta) = \infty.$$

There exists $g \subset \delta$ such that

(1) *g is $L[\tilde{E}]$-generic for a partial order*

$$\mathbb{P} = (\delta, <_{\mathbb{P}}) \in L[\tilde{E}]$$

which is δ-cc in $L[\tilde{E}]$.

(2) $\left(L[\tilde{E}][g]\right)^{<\delta} \subset L[\tilde{E}][g]$.

As a corollary to the two closure lemmas

Theorem

Suppose $\tilde{E}$ is a suitable extender sequence such that

$$o^{\tilde{E}}_{\mathrm{LONG}}(\delta) = \infty.$$

Suppose that for proper class of λ there exists an elementary embedding

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$$

with $\mathrm{CRT}(j) < \lambda$ such that $j(L_\lambda[\tilde{E}]) = L_\lambda[\tilde{E}]$.

- *Then in $L[\tilde{E}]$, for a proper class of λ there exists an elementary embedding,*

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1}).$$

proof:

Fix $\lambda > \delta$ and

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$$

with $\delta < \mathrm{CRT}(j) < \lambda$ such that $j(L_\lambda[\tilde{E}]) = L_\lambda[\tilde{E}]$.

Fix $g \subset \delta$ such that

1. *g is $L[\tilde{E}]$-generic for a partial order $(\delta, <_{\mathbb{P}}) \in L[\tilde{E}]$,*
2. *$\left(L[\tilde{E}][g]\right)^{<\delta} \subset L[\tilde{E}][g]$.*

By (2): $L[\tilde{E}][g] \cap V_{\lambda+1} \in L(V_{\lambda+1})$. Note that

$$j(L[\tilde{E}][g] \cap V_{\lambda+1}) = L[\tilde{E}][g] \cap V_{\lambda+1}.$$

Therefore j induces an elementary

$$j_g : (L(V_{\lambda+1}))^{L[\tilde{E}][g]} \to (L(V_{\lambda+1}))^{L[\tilde{E}][g]}.$$

Further by (2) and the closure $L[\tilde{E}]$ under extenders it follows that $j_g \in L[\tilde{E}][g]$.

proof continued ...

Thus we have shown that in a generic extension of $L[\tilde{E}]$; for a proper class of $\lambda > \delta$, there exists an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ with $\mathrm{CRT}(j) < \lambda$.

Theorem

The existence of a proper class of λ for which there is an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ with $\mathrm{CRT}(j) < \lambda$ is absolute between V and all generic extensions of V.

Using this theorem; in $L[\tilde{E}]$, for a proper class of λ there is an elementary embedding

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$$

with $\mathrm{CRT}(j) < \lambda$.

Doddages

A sequence

$$\tilde{\mathcal{E}} = \langle \mathcal{E}^\alpha_\beta : (\alpha, \beta) \in \mathrm{dom}(\tilde{\mathcal{E}}) \rangle$$

is a *Doddage* if for all $(\alpha, \beta) \in \mathrm{dom}(\tilde{\mathcal{E}})$, $\mathcal{E}^\alpha_\beta$ is a set of extenders such that

$$\mathrm{LTH}(E) = \alpha$$

for all $E \in \mathcal{E}^\alpha_\beta$.

Constructing from a Doddage: the inner model $L[\tilde{\mathcal{E}}]$

Suppose $\tilde{\mathcal{E}}$ is a Doddage then

$$L[\tilde{\mathcal{E}}] = L[\langle \cap \mathcal{E}^\alpha_\beta : (\alpha, \beta) \in \mathrm{dom}(\tilde{\mathcal{E}}) \rangle].$$

Martin-Steel Doddages

A Doddage,

$$\tilde{\mathcal{E}} = \langle \mathcal{E}^{\alpha}_{\beta} : (\alpha, \beta) \in \mathrm{dom}(\tilde{\mathcal{E}}) \rangle$$

is a *Martin-Steel Doddage* if the following hold for each pair

$$(\alpha, \beta) \in \mathrm{dom}(\tilde{E}).$$

Coherence Condition

For each extender $E \in \mathcal{E}^{\alpha}_{\beta}$ there exists an extender F such that

1. $\alpha < \mathrm{LTH}(F)$ *and* $E = F|\alpha$
2. $j_F(\tilde{\mathcal{E}})|(\alpha + 1, 0) = \tilde{\mathcal{E}}|(\alpha, \beta)$.

Novelty Condition

For all $\beta^* < \beta$, $(\alpha, \beta^*) \in \mathrm{dom}(\tilde{\mathcal{E}})$ and

$$E \cap L[\tilde{\mathcal{E}}|(\alpha, \beta)] \neq E^* \cap L[\tilde{\mathcal{E}}|(\alpha, \beta)]$$

for all $E \in \mathcal{E}^{\alpha}_{\beta}$ and for all $E^* \in \mathcal{E}^{\alpha}_{\beta^*}$

Initial Segment Condition

Suppose that $E \in \mathcal{E}^{\alpha}_{\beta}$ *and*

$$\kappa < \alpha^* < \alpha$$

where κ *is the critical point associated to* E*.*

Then there exists β^* *such that* $(\alpha^*, \beta^*) \in \mathrm{dom}(\tilde{\mathcal{E}})$ *and there exists* $E^* \in \mathcal{E}^{\alpha^*}_{\beta^*}$ *such that*

$$(E|\alpha^*) \cap L[\tilde{\mathcal{E}}|(\alpha^* + 1, 0)] = E^* \cap L[\tilde{\mathcal{E}}|(\alpha^* + 1, 0)].$$

Allowing long extenders one obtains generalized Martin-Steel Doddages.

Suitable Doddages

Definition

A suitable Doddage is [... a generalized Martin-Steel Doddage which contains no critical pairs ...]

Definition

Suppose $\tilde{\mathcal{E}}$ is a suitable Doddage of class length. Then

$$o^{\tilde{\mathcal{E}}}_{\mathrm{LONG}}(\delta) = \infty$$

if for all $\alpha > \delta$ there exists $(\alpha, \beta) \in \mathrm{dom}(\tilde{\mathcal{E}})$ and an extender $E \in \mathcal{E}^{\alpha}_{\beta}$ such that

$$j_E(\mathrm{CRT}(E)) = \delta.$$

Lemma

Suppose δ is supercompact. Then there is a suitable Doddage $\tilde{\mathcal{E}}$ such that

$$o^{\tilde{\mathcal{E}}}_{\text{LONG}}(\delta) = \infty$$

and such that $\tilde{\mathcal{E}}$ is definable from δ.

Definition

Suppose $\tilde{\mathcal{E}}$ is a Doddage. Then $\tilde{\mathcal{E}}$ is a *good Doddage* if for all $(\alpha, \beta) \in \mathrm{dom}(\tilde{\mathcal{E}})$

$$E \cap L[\tilde{\mathcal{E}}] = F \cap L[\tilde{\mathcal{E}}]$$

for all $E, F \in \mathcal{E}^{\alpha}_{\beta}$.

Theorem (Martin-Steel)

Assume IH and that $\tilde{\mathcal{E}}$ is a Martin-Steel Doddage. Then $\tilde{\mathcal{E}}$ is a good Doddage.

Lemma (Weak Covering)

Suppose that $\tilde{\mathcal{E}}$ is a good suitable Doddage such that

$$o^{\tilde{\mathcal{E}}}_{\mathrm{LONG}}(\delta) = \infty.$$

Suppose $\gamma > \delta$ and γ is a singular cardinal.

- *Then γ is singular in $L[\tilde{\mathcal{E}}]$ and*

$$\gamma^+ = (\gamma^+)^{L[\tilde{\mathcal{E}}]}.$$

The Doddage Conjecture

Theorem

Suppose there is a supercompact cardinal.

- *Then there is a suitable Doddage which is not good.*

The Doddage Conjecture at δ

Suppose δ is a supercompact cardinal. Then there exists a good suitable Doddage $\tilde{\mathcal{E}}$ such that

1. *$\tilde{\mathcal{E}}$ is Σ_3-definable from δ,*
2. $o^{\tilde{\mathcal{E}}}_{\mathrm{LONG}}(\delta) = \infty$.

Definition

Suppose that κ is an uncountable regular cardinal. Then κ is *ω-strongly measurable in* HOD if there exists $\gamma < \kappa$ such that:

(1) $(2^\gamma)^{\mathrm{HOD}} < \kappa$;

(2) There does not exist a sequence
$$\langle S_\alpha : \alpha < \gamma \rangle \in \mathrm{HOD}$$
of pairwise disjoint subsets of κ such that for each $\alpha < \gamma$, S_α is stationary in $\{\eta < \kappa \mid \mathrm{cof}(\eta) = \omega\}$.

The HOD Conjecture

There is a proper class of uncountable regular cardinals κ which are **not** *ω-strongly measurable in HOD.*

- **It is not known if there can exist even three cardinals which are ω-strongly measurable in HOD.**
- **It is not known if λ^+ can be ω-strongly measurable in HOD if λ is a singular strong limit with $\mathrm{cof}(\lambda) > \omega$.**

Equivalence Theorem

Suppose that δ is an extendible cardinal. Then the following are equivalent.

(1) *The Doddage Conjecture holds at δ.*

(2) *The HOD Conjecture holds.*

(3) *There exists a regular cardinal $\gamma > \delta$ which is not a measurable cardinal in HOD.*

An application

Theorem

Suppose that δ is an extendible cardinal and that there exists a cardinal $\gamma > \delta$ such that

$$\gamma^+ = (\gamma^+)^{\mathrm{HOD}}.$$

Then for all singular cardinals $\gamma > \delta$, $\gamma^+ = (\gamma^+)^{\mathrm{HOD}}$.

Proof: By the Equivalence Theorem, the Doddage Conjecture holds at δ. Therefore there is a good suitable Doddage $\tilde{\mathcal{E}}$ such that $\tilde{\mathcal{E}}$ is definable from δ and such that

$$o^{\tilde{\mathcal{E}}}_{\mathrm{LONG}}(\delta) = \infty.$$

Thus $L[\tilde{\mathcal{E}}] \subseteq \mathrm{HOD}$ and so by weak covering, for all singular cardinals $\gamma > \delta$, $\gamma^+ = (\gamma^+)^{\mathrm{HOD}}$. □

A second application

Theorem

Suppose that δ is an extendible cardinal and that there exists a regular cardinal $\gamma > \delta$ such that γ is not ω-strongly measurable in HOD.

- *Then there is no nontrivial elementary embedding*

$$j : \mathrm{HOD} \to \mathrm{HOD}$$

such that $\mathrm{CRT}(j) > \delta$.

L^{Ω}: an ultimate version of L?

- Mitchell and Steel defined the fine-structural version of extenders models up to the level of superstrong cardinals.
- Assuming IH and the existence of superstrong cardinals, Neeman and Steel proved existence up to the level of superstrong cardinals.
- If the Mitchell-Steel theory can be extended to the level of suitable extender sequences then modulo iteration hypotheses one would have an ultimate version of L: L^{Ω}.

The axiom $V = L^{\Omega}$ would be compatible with essentially all known large cardinal hypotheses.

- **The successful construction of L^{Ω} would complete the Jensen Program for finding the ultimate core model.**

Theorem

Suppose M is an iterable Mitchell-Steel model and that

$$M \models \mathrm{ZFC} + \text{“There is a Woodin cardinal”}.$$

- *Then M is a generic extension of an iterable Mitchell-Steel model, $N \subsetneq M$.* □

This theorem will generalize to L^Ω yielding as a corollary:

Theorem ($V = L^\Omega$)

Suppose there exists $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ with $\mathrm{CRT}(j) < \lambda$.

- *Then in $L(V_{\lambda+1})$ the club filter of λ^+ is not an ultrafilter on each cofinality.*

- This significantly constrains the possible theories for $L(V_{\lambda+1})$
  - something is wrong.

Theorem

Suppose that $\Gamma \subset \mathcal{P}(\mathbb{R})$, $\Gamma = L(\Gamma, \mathbb{R}) \cap \mathcal{P}(\mathbb{R})$, *and*

$$L(\Gamma, \mathbb{R}) \models \mathrm{AD}_{\mathbb{R}} + \text{“}\Theta \text{ is regular”}.$$

Let $N = (\mathrm{HOD})^{L(\Gamma,\mathbb{R})} \cap V_\Theta$ *where* $\Theta = \Theta^{L(\Gamma,\mathbb{R})}$.

- *Then* N *is not a generic extension of any inner model* $M \subsetneq N$. □

Much more is true:

- N is ordinal definable in any set extension of N.
- If $N[g] = M[G]$ then $N \subseteq M$.

L_S^Ω: an alternative to L^Ω?

Claim

If L^Ω exists then so does $L^\Omega[\mathcal{S}_0]$ where $\mathcal{S}_0$ is the iteration strategy of L^Ω restricted to the least Woodin cardinal of L^Ω.

This suggests the possibility that L_S^Ω exists where L_S^Ω is the version of L^Ω where one adds the entire transfinite iteration strategy of L_S^Ω.

- I believed that I could prove using AD^+-theory that L_S^Ω was inconsistent with existence of cardinals strong past Woodin cardinals (witnessed by extenders on the sequence).
  - **This would imply that L_S^Ω cannot exist.**

Conjecture

Assume there is a proper class of supercompact cardinals.

- *Then L_S^Ω exists.*

The axiom: $V = L_S^\Omega$

1. *There is a supercompact cardinal.*
2. *There exist a universally Baire set $A \subset \mathbb{R}$ and $\gamma < \Theta^{L(A,\mathbb{R})}$ such that*

$$V \equiv (\mathrm{HOD})^{L(A,\mathbb{R})} \cap V_\gamma$$

for all Π_2-sentences.

Reference: Suitable Extender Sequences, preprint July 2009