

PRODUCTS OF FINITE SETS WITH
SUBMEASURES

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Product of finite sets.

Theorem. If $r \in \omega$, $k \in \omega$, and $n_i : i \in k$ are natural numbers, then there are numbers $m_i : i \in k$ such that for every partition of the product $\prod_i m_i$ into r many parts, one of them contains a subset of the form $\prod_i b_i$, where $|b_i| \geq n_i$.

Infinite product.

Theorem. (DiPrisco, Henle) If $r \in \omega$ and $n_i : i \in \omega$ are natural numbers then there are natural numbers $m_i : i \in \omega$ such that for every partition of the product $\prod_i m_i$ into r many Borel pieces, one of the pieces contains a subset of the form $\prod_i b_i$, where $|b_i| \geq n_i$.

The proof uses a partition property of $[\omega]^{\aleph_0}$. Later, an upper bound for m_i 's has been found in terms of the Ackermann function.

Parametrized infinite product

Theorem. (DiPrisco, Llopis, Todorćević) If $r \in \omega$ and $n_i : i \in \omega$ are natural numbers then there are natural numbers $m_i : i \in \omega$ such that for every partition of the product $\prod_i m_i \times \omega$ into r many Borel pieces, one of the pieces contains a subset of the form $\prod_i b_i \times c$, where $|b_i| \geq n_i$ and $c \subset \omega$ is infinite.

The proof does not provide an upper bound for the numbers m_i .

Parametrized infinite product with submeasures

Theorem. If $r \in \omega$ and $n_i : i \in \omega$ are natural numbers then there are natural numbers $m_i : i \in \omega$ and finite sets $a_i : i \in \omega$ such that for every sequence $\phi_i : i \in \omega$ of submeasures on a_i such that $\phi_i(a_i) > m_i$ and every partition of the product $\prod_i a_i \times \omega$ into r many Borel pieces, one of the pieces contains a subset of the form $\prod_i b_i \times c$, where $\phi_i(b_i) \geq n_i$ and $c \subset \omega$ is infinite.

The proof uses creature forcing. It provides an approximately double exponential bound on the growth of the numbers m_i and sets a_i .

Parametrization with a measure

Theorem. If $\varepsilon > 0$ is real and $n_i : i \in \omega$ are natural numbers then there are natural numbers $m_i : i \in \omega$ and finite sets $a_i : i \in \omega$ such that for every sequence $\phi_i : i \in \omega$ of *measures* on a_i such that $\phi_i(a_i) > m_i$ and every Borel set $B \subset \prod_i a_i \times \omega \times [0, 1]$ with vertical sections of Lebesgue mass at least ε , B contains a subset of the form $\prod_i b_i \times c \times \{z\}$ where $\phi_i(b_i) > n_i$ and $c \subset \omega$ is infinite.

An application.

Suppose $P_i : i \in \omega$ are forcings obtained from σ -ideals generated by a compact collection of compact sets. (limsup infinity forcings). Then the product $\prod_i P_i$ does not add a splitting real.

This strengthens the result of Laver on product of Sacks forcing.

Creature forcing.

A *setup* is an atomic partial order C with a finite set of atoms and a norm function to the reals, preserving the order and vanishing at the atoms.

If $C_i : i \in \omega$ are setups then the derived creature forcing P is just the subset of $\prod_i C_i$ consisting of functions on which the norms tend to infinity.

Main point. Under suitable conditions on the setups C_i the creature forcing is proper, bounding, and adds no splitting reals.

The creatures we need.

Let a be finite, ϕ a submeasure on it. A *creature* (nonatomic element of the setup) is a pair $\langle b, r \rangle$ where $b \subset a$ and $r \geq \log(\phi(b) + 1)$. Define

- $norm(b, r) = \varepsilon \log(\log(\phi(b) + 1) - r + 1)$
- $\langle c, s \rangle \leq \langle b, r \rangle$ if $c \subset b$ and $s \geq r$.