Covering reals by translations of a compact set

Tomek Bartoszynski (joint work with S. Shelah)

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**Problem (Gruenhage):** Given a compact subset of the real line $K$ is it consistent that the real line is covered by $< 2^{\aleph_0}$ translations of $K$? Or more generally, if $K$ is a compact subset of a Polish group.

**Obstruction:** there exists a perfect set $P$ such that for every $x$, $(K + x) \cap P$ is countable.

This obstruction is a $\Sigma^1_2$ property of $K$.
To see this note that $K$ is small iff $\exists P$

1. $P$ is closed and uncountable ($\Sigma^1_1$),
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If $C$ is the ordinary Cantor set then $\mathbb{R}$ is not covered by $< 2^{\aleph_0}$ translations of $C$. (Gruenhage)

if $C$ has packing dimension $< 1$ then $\mathbb{R}$ is not covered by $< 2^{\aleph_0}$ translations of $C$. (Darji-Keleti)

if $K$ is not meager then $\mathbb{R}$ is covered by countably many translations of $K$ (folklore),

if $K$ has positive measure then $\mathbb{R}$ is covered by non$(\mathcal{N})$ translations of $K$ (folklore),

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Let $G$ be a Polish group and let $\text{cov}^*_G(M)$ be the minimal cardinality of set $X \subseteq G$ such that for some closed nowhere dense set $M$, $X + M = G$. The value of $\text{cov}^*_G(M)$ depends on $G$. (Miller-Steprans)

Suppose that given an uncountable set $X \subseteq \mathbb{R}$ we can find a compact measure zero set $K$ such that $\bigcup_{x \in X}(K + x) = \mathbb{R}$, then Borel Conjecture + Dual Borel Conjecture holds.
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Definition

A perfect set $K \subseteq 2^\omega$ is big if for every $n \in \omega$ there exists $j_n \in \omega$ such that for $X \subseteq 2^\omega$ and $x \in 2^\omega$, if

1. $|X| \leq n$,
2. $(2^\omega \setminus K) + X \neq 2^\omega$,
3. $x \upharpoonright j_n \in X \upharpoonright j_n$,

then

$$(2^\omega \setminus K) + (X \cup \{x\}) \neq 2^\omega.$$  

We say that $K$ is big* if $K \cap [s]$ is big for every $s \in 2^{<\omega}$ such that $K \cap [s] \neq \emptyset$.

Lemma

If $K$ is big then $K$ is not small.
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*If $K$ is big then $K$ is not small.*
Theorem

If $K$ is big$^*$, then there is a ccc-extension of the universe in which $2^\omega$ is covered by $< 2^{\aleph_0}$ translations of $K$.

Let $Q = \{ q \in 2^\omega : \forall \infty n \ q(n) = 0 \}$.

Lemma

Suppose that $K \subseteq 2^\omega$ is big$^*$. There exists a ccc forcing notion $P_K$ which adds real $z_K \in 2^\omega$ such that

$$\forces_{P_K} \forall x \in 2^\omega \cap V \exists q \in Q \ x \in K + z_K + q.$$ 

Let $P_K$ be the collection of pairs $(t, X)$ such that

1. $t \in 2^{<\omega}$ and $X$ is a finite subset of $2^\omega$,
2. $((2^\omega \setminus K) + X) \cap [t] \neq \emptyset$.

For $(t_0, X_0), (t_1, X_1) \in P_K$, we put $(t_1, X_1) \geq (t_0, X_0)$ if $t_0 \subseteq t_1$ and $X_0 \subseteq X_1$. 

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If $K$ is big*, then there is a ccc-extension of the universe in which $2^\omega$ is covered by $< 2^{\aleph_0}$ translations of $K$.

Let $\mathcal{Q} = \{ q \in 2^\omega : \forall n q(n) = 0 \}$.

Lemma

Suppose that $K \subseteq 2^\omega$ is big*. There exists a ccc forcing notion $\mathbb{P}_K$ which adds real $z_K \in 2^\omega$ such that

$$\vdash_{\mathbb{P}_K} \forall x \in 2^\omega \cap V \exists q \in \mathcal{Q} x \in K + z_K + q.$$ 

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If $K$ is big$^\star$, then there is a ccc-extension of the universe in which $2^\omega$ is covered by $< 2^{\aleph_0}$ translations of $K$.

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Theorem

Suppose that $K \subseteq 2^\omega$. If $K$ is not small, then it is consistent that $2^\omega$ is covered by $< 2^{\aleph_0}$ translations of $K$.

The same holds for subsets $K$ of locally compact abelian Polish groups.

If $K$ is not small then in the Sacks model $V^{S_\omega_2}$,

$$\forall x \in 2^\omega \ \exists z \in V \cap 2^\omega \ x \in K + z.$$ 

The following is a technical restatement of this fact.

Theorem

Suppose that $p \Vdash_{S_\omega_2} \check{x} \in 2^\omega \setminus V$. Then there exists $p' \geq p$ and a perfect set $P \subseteq 2^\omega$ such that for every perfect set $Q \subseteq P$ there exists $q \geq p'$ such that $q \Vdash \check{x} \in Q$.

Suppose that $K$ is not small and let $p \Vdash_{S_\omega_2} \check{x} \in 2^\omega$. Find $p' \geq p$ and $P$. Since $K$ is not small there is $z \in 2^\omega$ such that $P \cap (K + z)$ is uncountable. Let $Q \subseteq P \cap (K + z)$ be a perfect set. It follows that there is $q \geq p'$ such that $q \Vdash_{S_\omega_2} \check{x} \in Q \subseteq K + z$. 

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Suppose that $K$ is not small and let $p \vDash S_{\omega_2} \ x \in 2^\omega$. Find $p' \geq p$ and $P$. Since $K$ is not small there is $z \in 2^\omega$ such that $P \cap (K + z)$ is uncountable. Let $Q \subseteq P \cap (K + z)$ be a perfect set. It follows that there is $q \geq p'$ such that $q \vDash \dot{x} \in Q \subset K + z$.

Tomek Bartoszynski (joint work with S. Shelah) Covering reals by translations of a compact set
Theorem

Suppose that $K \subseteq 2^\omega$. If $K$ is not small, then it is consistent that $2^\omega$ is covered by $< 2^{\aleph_0}$ translations of $K$.

The same holds for subsets $K$ of locally compact abelian Polish groups.

If $K$ is not small then in the Sacks model $V^{S_{\omega_1}}$, 

$$\forall x \in 2^\omega \ \exists z \in V \cap 2^\omega \ x \in K + z.$$ 

The following is a technical restatement of this fact.

Theorem

Suppose that $p \Vdash_{S_{\omega_2}} \dot{x} \in 2^\omega \setminus V$. Then there exists $p' \geq p$ and a perfect set $P \subseteq 2^\omega$ such that for every perfect set $Q \subseteq P$ there exists $q \geq p'$ such that $q \Vdash \dot{x} \in Q$.

Suppose that $K$ is not small and let $p \Vdash_{S_{\omega_2}} \dot{x} \in 2^\omega$. Find $p' \geq p$ and $P$. Since $K$ is not small there is $z \in 2^\omega$ such that $P \cap (K + z)$ is uncountable. Let $Q \subseteq P \cap (K + z)$ be a perfect set. It follows that there is $q \geq p'$ such that $q \Vdash_{S_{\omega_2}} \dot{x} \in Q \subseteq K + z$. 

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Covering reals by translations of a compact set
\[ \text{CPA} \iff 2^\mathbb{N}_0 = \aleph_2 \text{ and for every "appropriately" dense family } \mathcal{E}_0 \subset S \text{ there is an } \mathcal{E}_0 \subset \mathcal{E} \text{ such that } |\mathcal{E}_0| \leq \aleph_1 \text{ and } |\mathbb{R} \setminus \bigcup \mathcal{E}_0| \leq \aleph_1. \]

**Theorem**

Assume \( \text{CPA}_{\text{prism}} \). Then if \( K \) is not small then \( \mathbb{R} \) is covered by \( \aleph_1 \) translations of \( K \).
Examples of sets which are not small

Let \( \{ I_n : n \in \omega \} \) be a partition of \( \omega \) into finite sets of increasing size and let \( K_n \subset 2^{I_n} \). Consider sets of form \( K = \prod_n K_n \).

**Lemma**

If \( \lim_n \frac{|K_n|}{2^{|I_n|}} = 1 \) then \( K \) is big*.

**Lemma (Elekes-Toth)**

Suppose that \( I \subset \omega \) is finite, \( n \in \omega \) and \( C \subset 2^I \) is such that

\[
\frac{|C|}{2^{|I|}} \geq 1 - \frac{1}{n+1}.
\]

For any \( X \subset 2^I \) of size \( \leq n \) there exists \( t \in 2^I \) such that \( t + X \subseteq C \).

Choose \( K_n \)'s such that

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1 - \frac{1}{n+1} \leq \frac{|K_n|}{2^{|I_n|}} \leq 1 - \frac{1}{2n+1}.
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Then \( K = \prod_n K_n \) has measure zero.

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Then \( K = \prod_n K_n \) has measure zero.
If \( \lim_{n} \frac{|K_n|}{|2^I_n|} < 1 \) then \( K \) may be small or big\(^*\), depending on the choice of \( K_n \)'s.

**Lemma**

Fix \( \varepsilon > 0 \). There exists \( K_n \subseteq 2^I_n \) such that for each \( n \),
\[
|K_n|/2^{|I_n|} \leq \varepsilon \quad \text{and} \quad K = \prod_n K_n \text{ is small}.
\]

**Lemma**

Fix a sequence of positive reals \( \{\varepsilon_n : n \in \omega\} \). There exists a sequence \( K_n \subseteq 2^I_n \) such that for each \( n \),
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Theorem (Bartoszynski-Shelah)

Suppose that $m \in \omega$ and $0 < \delta < \varepsilon < 1$ are given. There exists $n \in \omega$ such that for every finite set $I \subseteq \omega$ of size at least $n$, there exists a set $C \subseteq 2^I$ such that $\varepsilon + \delta \geq |C| \cdot 2^{-|I|} \geq \varepsilon - \delta$ and for every set $X \subseteq 2^I$, $|X| \leq m$

\[
\left| \frac{|\bigcap_{s \in X} (C + s)|}{2^{|I|}} - \varepsilon |X| \right| < \delta.
\]

Note that the theorem says that we can choose $C$ is such a way that for any sequences $s_1, \ldots, s_m \in 2^I$ the sets $s_1 + C, \ldots, s_m + C$ are probabilistically independent with error $\delta$. Thus, if we choose $\delta$ to be much smaller than $\varepsilon^m$, then if $|X| < m$ it follows that $\bigcap_{s \in X} (C + s) \neq \emptyset$. In particular, if $t \in \bigcap_{s \in X} (C + s)$ then $X \subseteq C + t$. 

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