The descriptive set theory of orbit equivalence

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Suppose that $\mathcal{A}$ is a standard Borel space and $E$ is an equivalence relation on $\mathcal{A}$.

**Definition**

$E$ on $\mathcal{A}$ is **Borel reducible** to $F$ on $\mathcal{B}$, denoted $E \leq_B F$ if there is a Borel map $\phi: \mathcal{A} \to \mathcal{B}$ such that

$$xEy \iff \phi(x)F\phi(y).$$

This is meant to reflect that $F$ is “more complex” than $E$ and that the points of $\mathcal{A}$ can be classified up to $E$-equivalence by a Borel assignment of invariants that are $F$-equivalence classes.
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\( E \) is smooth if \( E \leq_B id(\mathbb{R}) \).
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$E$ is smooth if $E \leq_B id(\mathbb{R})$.

Example: If we consider $E$ to be the equivalence relation of similarity on the space of $n \times n$ matrices, then we can let $f(A)$ be the Jordan form of $A$. 
$E_0$ is the equivalence relation given by eventual agreement on $2^\mathbb{N}$:

$$xE_0y \iff \exists m \in \mathbb{N} \quad \forall n > m \quad x(n) = y(n)$$

$E$ is hyperfinite if $E \leq_B E_0$, or, equivalently, $E$ is given by a Borel action of $\mathbb{Z}$.
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$$id(1) <_B id(2) <_B \ldots <_B id(\mathbb{N}) <_B id(\mathbb{R}) <_B E_0$$
Countable structures

Suppose that $\mathcal{L}$ is a countable relational language and $\text{Mod}(\mathcal{L})$ is the set of models for $\mathcal{L}$ with the underlying set $\mathbb{N}$. 
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Then each relation $R_i \in \mathcal{L}$ is a subset of $\mathbb{N}^{a(i)}$ where $a(i)$ is the parity of $R_i$.

Thus, $\text{Mod}(\mathcal{L})$ can be identified with $\prod_{i \in \mathbb{N}} 2^{\mathbb{N}^{a(i)}}$ or $2^{\mathbb{N}}$ and equip this space with the product topology to make $\text{Mod}(\mathcal{L})$ a standard Borel space.
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The logic action of $S_\infty$ on $\text{Mod}(\mathcal{L})$ is given by

$$f \cdot M \models \phi(a_1, \ldots, a_n) \iff M \models \phi(f^{-1}(a_1), \ldots, f^{-1}(a_n)).$$

The orbit equivalence relation of $S_\infty$ on $\text{Mod}(\mathcal{L})$ gives rise to the isomorphism equivalence relation.
### Definition

An equivalence relation $E$ on a Borel space $X$ is **classifiable by countable structures** if there is a countable language $\mathcal{L}$ and a Borel map $\phi : X \to Mod(\mathcal{L})$ such that

$$xEy \iff \phi(x) \cong \phi(y).$$
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The equivalence relations that are classifiable by countable structures include equivalence relations that can be reasonably classified using countable groups, graphs, fields, etc. as complete invariants.
The space of actions

Γ is a countable infinite group and \((X, \mu)\) a standard probability space (Borel isomorphic to \([0, 1]\) with Lebesgue measure.) \(\Gamma \curvearrowright (X, \mu)\) by Borel automorphisms. This gives rise to the orbit equivalence relation

\[ E_\Gamma = \{(x, \gamma \cdot x) \mid x \in X, \gamma \in \Gamma\}. \]

The action is:

- **free** if for any \(\gamma \in \Gamma\), \(\gamma \cdot x = x \implies \gamma = e\).
- **measure preserving** if for any Borel \(A \subset X\) and \(\gamma \in \Gamma\), \(\mu(A) = \mu(\gamma \cdot A)\).
- **ergodic** if for any Borel \(\Gamma\)-invariant \(A \subset X\), \(\mu(A) = 1\) or \(\mu(A) = 0\).
The space of actions

For a group $\Gamma$, $A(\Gamma, X, \mu)$ is the space of measure preserving actions of $\Gamma$ on $(X, \mu)$ where $(X, \mu)$ is a standard probability space. This identifies with the space of homomorphisms of $\Gamma$ into $Aut(X, \mu)$. 
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$Aut(X, \mu)$ is a Polish space with the weak topology generated by the functions

$$A \mapsto T(A) \quad A \in MALG_\mu, T \in Aut(X, \mu)$$

and $Aut(X, \mu)^\Gamma$ is Polish with the product topology.
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The space of homomorphisms of $\Gamma$ into $Aut(X, \mu)$ is a closed subset of $Aut(X, \mu)^{\Gamma}$. The space of free and ergodic actions, which will be denoted $A_{\Gamma}$ is closed in $A(\Gamma, X, \mu)$. 
Orbit equivalence

**Definition**

Two actions $\Gamma \curvearrowright (X, \mu)$, $\Delta \curvearrowright (Y, \nu)$ are **orbit equivalent** if there are conull subsets $A \subset X$, $B \subset Y$ and a measure-preserving bijection $\phi: A \rightarrow B$ such that for all $x \in A$,

$$\phi(\Gamma \cdot x) = \Delta \cdot \phi(x).$$
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The **group measure space construction** (Murray-von Neumann, 1936) associates to every free, measure preserving, ergodic action $\Gamma \curvearrowright (X, \mu)$ a $\text{II}_1$ factor $L^\infty(X) \rtimes \Gamma$. 
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Theorem (Feldman-Moore, 1977)

A measure space isomorphism $\phi : X \to Y$ extends to $L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Delta$ iff $\phi$ is an orbit equivalence.
**Orbit equivalence**

**Definition**

$OE_\Gamma$ is the equivalence relation on $\mathcal{A}_\Gamma$ given by two actions being orbit equivalent.

**Question**

For a given group $\Gamma$, how many orbit-inequivalent actions that are free, measure preserving, ergodic does $\Gamma$ admit?

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Amenable groups

Theorem (Dye, about 1960)
Any two measure preserving ergodic actions of \( \mathbb{Z} \) are orbit equivalent.
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Any two measure preserving ergodic actions of $\mathbb{Z}$ are orbit equivalent.

Theorem (Ornstein-Weiss, 1980)
Any two measure preserving ergodic actions of an amenable group are orbit equivalent to such an action of $\mathbb{Z}$. 
Almost invariant vectors

Let $\pi: \Gamma \to U(H)$ be a unitary representation of $\Gamma$ on some Hilbert space $H$.
Then $v \in H$ is a $\pi$-invariant vector if for all $\gamma \in \Gamma$, $\pi(\gamma) \cdot v = v$.
Almost invariant vectors

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Then \( v \in H \) is a \( \pi \)-invariant vector if for all \( \gamma \in \Gamma \), \( \pi(\gamma) \cdot v = v \).

**Definition**

\( \pi \) admits almost invariant vectors if for any \( Q \subset \Gamma \) finite, \( \epsilon > 0 \), there is a unit vector \( v \in H \), such that

\[
\forall \gamma \in Q \quad \| \pi(\gamma) \cdot v - v \| < \epsilon.
\]
Amenable groups

Γ acts on $l^2(\Gamma)$ by shift

$$\gamma \cdot f(x) = f(\gamma^{-1} \cdot x).$$

**Definition**

Γ is **amenable** if the representation of Γ on $l^2(\Gamma)$ obtained from the shift action admits almost invariant vectors.
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Γ is **amenable** if the representation of Γ on $l^2(\Gamma)$ obtained from the shift action admits almost invariant vectors.

Equivalently, if for every $\epsilon > 0$, $A \subset \Gamma$ finite, there is a finite set $F \subset \Gamma$ such that

$$|\gamma \cdot A \Delta A| < \epsilon \cdot |A| \quad \forall \gamma \in A.$$
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Examples: finite groups, \( \mathbb{Z} \), abelian groups

Non-example: \( F_2 \)
Non-amenable groups

<table>
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<th>Theorem (Connes-Weiss, Schmidt, 1980)</th>
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Non-amenable groups

Theorem (Connes-Weiss, Schmidt, 1980)
If $\Gamma$ does not have property (T), then $\Gamma$ admits at least 2 orbit inequivalent free, measure preserving, ergodic actions.

Theorem (Bezuglyï - Golodets, 1981)
There is some countable infinite group that admits continuum many orbit inequivalent free, measure preserving, ergodic actions.
**Property (T)**

**Definition**

Γ has property (T) if there is a finite $Q \subset \Gamma$ and $\epsilon > 0$ such that for any unitary representation $\pi$ of $\Gamma$, if $\pi$ admits a $(Q, \epsilon)$-invariant vector, then $\pi$ admits an invariant unit vector.

Example: $SL_3(\mathbb{Z})$
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Some groups with property (T) have continuum many orbit inequivalent free, measure preserving, ergodic actions. $SL_3(\mathbb{Z})$ is among these groups.

**Theorem (Hjorth, 2005)**

All groups with property (T) admit continuum many orbit inequivalent free, measure preserving, ergodic actions.
**Relative property (T)**

**Definition**

If $\Delta \leq \Gamma$, then the pair $(\Gamma, \Delta)$ has **relative property (T)** if there is a finite $Q \subset \Gamma$ and $\epsilon > 0$ such that for any unitary representation $\pi$ of $\Gamma$, if $\pi$ admits a $(Q, \epsilon)$-invariant vector, then $\pi|\Delta$ admits an invariant unit vector.
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If we consider the usual action of $SL_2(\mathbb{Z})$ on $\mathbb{Z}^2$ by matrix multiplication, then the pair $(SL_2(\mathbb{Z}) \rtimes \mathbb{Z}^2, \mathbb{Z}^2)$ has relative property (T).

$F_n$ embeds into $SL_2(\mathbb{Z})$ as a finite index subgroup to induce an action of $F_n$ on $\mathbb{Z}^2$. The pair $(F_n \rtimes \mathbb{Z}^2, \mathbb{Z}^2)$ also has relative property (T).
$SL_2(\mathbb{Z})$ also acts on $(\mathbb{T}^2, h)$ where $h$ is the Haar measure. We may identify $\mathbb{Z}^2$ with the group of characters on $\mathbb{T}^2$. 
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**Theorem (Gaboriau - Popa, 2006)**

$F_n$ for $n \geq 2$ admits continuum many orbit inequivalent free, measure preserving, ergodic actions.
The following classes of non-amenable groups also admit continuum many orbit inequivalent actions:

- Weakly rigid groups (Popa, 2006);
- Products of groups satisfying a certain cohomological property (Monod-Shalom, 2006);
- Mapping class groups (Kida, 2007).
Relative property (T) of $F_2$

For $\Delta \leq \Gamma$ and an action $\Delta \curvearrowright (Z, \nu)$, one can induce an action of $\Gamma$ on the space $(Z^N, \nu^N)$ where $N = [\Gamma : \Delta]$.
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**Theorem (Ioana)**

If \( F_2 \leq \Gamma \), then \( \Gamma \) admits continuum many orbit inequivalent free, measure preserving, ergodic actions.
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**Theorem (Ioana)**

If $\mathbb{F}_2 \leq \Gamma$, then $\Gamma$ admits continuum many orbit inequivalent free, measure preserving, ergodic actions.

There are non-amenable groups that don’t contain a copy of $\mathbb{F}_2$ (Ol’šanskiĭ, 1980).
Suppose $\Gamma \curvearrowright^{a_0} (X, \mu)$ and $\Delta \curvearrowright^{b_0} (X, \mu)$ such that $E^{b_0}_{\Delta} \subset E^{a_0}_{\Gamma}$. Given $\Delta \curvearrowright^{a} (Z, \nu)$, we produce a way to induce actions $\Delta \curvearrowright^{c} (Y, m)$ and $\Gamma \curvearrowright^{d} (Y, m)$.
Suppose $\Gamma \curvearrowright^{a_0} (X, \mu)$ and $\Delta \curvearrowright^{b_0} (X, \mu)$ such that $E_{\Delta}^{b_0} \subset E_{\Gamma}^{a_0}$. Given $\Delta \curvearrowright^{a} (Z, \nu)$, we produce a way to induce actions $\Delta \curvearrowright^{c} (Y, m)$ and $\Gamma \curvearrowright^{d} (Y, m)$.

**Theorem (E.)**

Suppose that there exist free, measure preserving actions $\Gamma \curvearrowright (X, \mu)$, $F_2 \curvearrowright (X, \mu)$ such that $\Gamma$ acts ergodically and $E_{F_2} \subset E_{\Gamma}$. Then $\Gamma$ admits continuum many orbit inequivalent free, measure preserving ergodic actions.
Theorem (Gaboriau-Lyons)
Every non-amenable group admits a free, measure preserving, ergodic action whose orbit equivalence relation contains a subequivalence relation given by a free, measure preserving action of $F_2$. 

Corollary
Every non-amenable group admits continuum many orbit inequivalent free, measure preserving, ergodic actions.
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Corollary

Every non-amenable group admits continuum many orbit inequivalent free, measure preserving, ergodic actions.
\( \mathbf{F}_2 \) admits continuum many non-isomorphic irreducible representations. These can be turned into actions of \( \mathbf{F}_2 \ltimes (\mathbb{Z}_i, \nu_i) \).

\( \mathbf{F}_2 \) also acts on \((\mathbb{T}^2, h)\) where \( h \) is the Haar measure. Then \( \hat{\mathbb{T}}^2 \cong \mathbb{Z}^2 \) and \( (\mathbf{F}_2 \ltimes \mathbb{Z}^2, \mathbb{Z}^2) \) has relative property \( T \).
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Using the actions of $F_2$ and $\Gamma$ from the theorem of Gaboriau and Lyons, we induce from the diagonal action $F_2 \rtimes^a (Z_i \times T^2, \nu_i \times h)$. 
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Using the actions of $\mathbb{F}_2$ and $\Gamma$ from the theorem of Gaboriau and Lyons, we induce from the diagonal action $\mathbb{F}_2 \rtimes^a (\mathbb{Z}_i \times \mathbb{T}^2, \nu_i \times h)$.

Of these induced actions, only countably many will be orbit equivalent to each other.
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Non-classification by countable structures

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\( E_0 \) can be reduced to \( OE_\Gamma \) and \( OE_\Gamma \) is not classifiable by countable structures:

- \( \Gamma = \mathbb{F}_n \) or \( \Gamma \) has Property(T) (Tornquist);
- \( \Gamma \) has a copy of \( \mathbb{F}_2 \) (Ioana-Kechris);
- \( \Gamma \) is non-amenable (E-Ioana-Kechris-Tsankov).

\( OE_\Gamma \) is very complex. It is impossible to assign a real valued (or countable algebraic groups, etc) invariant to \( OE_\Gamma \).
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Hjorth showed that the action of \( U(H) \) by conjugacy on the irreducible unitary representations of \( F_2 \) on \( H \) is turbulent.
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2. For any countable language $\mathcal{L}$, isomorphism on $\text{Mod}(\mathcal{L})$ is Borel reducible to isomorphism of factors;

3. Isomorphism of factors is Borel reducible to an equivalence relation arising from a continuous action of the unitary group of $l^2(\mathbb{N})$ on a Polish space. As a result, it is not the case that every analytic equivalence relation Borel reduces to isomorphism of factors.
Inducing an action of $\Gamma$

Let $\Gamma \curvearrowright^{a_0} (X, \mu)$ be free, measure preserving, ergodic and let $\Delta \curvearrowright^{b_0} (X, \mu)$, $\Delta \curvearrowright^{a} (Z, \nu)$ be free, measure preserving such that

$$E_{\Delta}^{b_0} \subset E_{\Gamma}^{a_0}.$$
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**Definition**

\[ Y = \{(x, f) \mid f: [x]_{\Gamma} \to Z, \quad f(\gamma_0 \cdot x) = \gamma_0 \cdot f(x) \quad \forall \gamma_0 \in \Delta\}. \]
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\[ Y = \{(x, f) \mid f : [x]_{\Gamma} \to Z, \quad f(\gamma_0 \cdot x) = \gamma_0 \cdot f(x) \quad \forall \gamma_0 \in \Delta\}. \]

We may assume that every $E_{\Gamma}$ equivalence class contains infinitely many $E_{\Delta}$ equivalence classes (the number of equivalence classes is constant since $E_{\Gamma}$ is ergodic).

$Y$ will be represented as $(X \times Z^\mathbb{N}, \mu \times \nu^\mathbb{N})$. 
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Given $(x, f)$, $f$ can be represented by choosing one value for each $\Delta$-equivalence class in $[x]_\Gamma$. 
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Given $(x, f)$, $f$ can be represented by choosing one value for each $\Delta$-equivalence class in $[x]_\Gamma$.

There exists a Borel sequence of functions $\{g_i : X \to X\}_{i \in \mathbb{N}}$ such that

- $g_0(x) = x$ for every $x \in X$
- given $x \in X$, $\{g_i(x)\}_{i \in \mathbb{N}}$ enumerates a transversal for the $\Delta$-equivalence classes in $[x]_\Gamma$;
- $g_i(x) \neq g_j(x)$ for $x \in X$ and $i \neq j$. 
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- $g_i(x) \neq g_j(x)$ for $x \in X$ and $i \neq j$.

Then identify $(x, f)$ with

$$(x, f(g_0(x)), f(g_1(x)), \ldots) \in (X \times \mathbb{Z}^\mathbb{N}, \mu \times \nu^\mathbb{N}).$$
Inducing an action of $\Gamma$

Let $\Gamma \curvearrowright Y$ by

$$\gamma \cdot (x, f) = (\gamma \cdot x, f).$$

The action of $\Gamma$ on a point $(x, f)$ rearranges the representation of $f$ in $(\mathbb{Z}^N, \nu^N)$. 
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This is given by a cocycle $\beta : \Gamma \times X \to S_\infty \ltimes \Delta^\mathbb{N}$. So then $\Gamma \acts^c (X \times \mathbb{Z}^\mathbb{N}, \mu \times \nu^\mathbb{N})$ by

$$\gamma \cdot (x, f) = (\gamma \cdot x, \beta(\gamma, x) \cdot f)$$

where

$S_\infty \ltimes \Delta^\mathbb{N} \acts \mathbb{Z}^\mathbb{N}$ by

$$(\alpha, \delta) \cdot f(k) = \delta(k) \cdot f(\alpha^{-1}(k)).$$
Actions of $\Gamma$ and $\Delta$

$\Gamma \curvearrowright^c (X \times \mathbb{Z}^N, \mu \times \nu^N)$ is free, measure preserving.
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$\Gamma \acts c (X \times \mathbb{Z}^\mathbb{N}, \mu \times \nu^\mathbb{N})$ is free, measure preserving.

Consider the inclusion cocycle $\sigma: \Delta \times X \to \Gamma$ given by

$$\sigma(\delta, x) = \gamma \iff \delta \cdot x = \gamma \cdot x.$$  

We also get a free, measure preserving action $\Delta \acts d (X \times \mathbb{Z}^\mathbb{N}, \mu \times \nu^\mathbb{N})$ by letting

$$\delta \cdot (x, f) = \sigma(\delta, x) \cdot (x, f).$$
Actions of $\Gamma$ and $\Delta$

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We obtain a probability measure $m$ on $Y$ that is ergodic with respect to the action $c$ of $\Gamma$ by taking an ergodic decomposition.
Lemma

Let $F_2 \curvearrowright^a (\mathbb{T}^2, h)$ and $F_2 \curvearrowright^{a_\pi} (Z, \nu)$ be free, measure preserving, weakly mixing. Then there are actions $\Gamma \curvearrowright^c (Y, m)$ and $F_2 \curvearrowright^d (Y, m)$ such that:

1. $\Gamma \curvearrowright^c (Y, m)$ is free, measure preserving, ergodic;
2. $F_2 \curvearrowright^d (Y, m)$ is free, measure preserving;
3. $E_{F_2}^d \subset E_{\Gamma}^c$;
4. for any non non-null $d$-invariant subset $Y_0 \subset Y$, $a \times a_\pi$ is a factor of $d|Y_0$;
5. There is an $F_2$-equivariant, measure preserving, surjective map $q: Y \rightarrow \mathbb{T}^2$ such that $\forall \gamma \in \Gamma \setminus \{e\}$,

$$m(\{y \in Y \mid q(\gamma c \cdot y) = q(y)\}) = 0.$$
The Koopman representation

**Definition**

When $\Gamma \curvearrowright^a (X, \mu)$, the **Koopman representation**, $\kappa_0^a$, is given by

$$\gamma \cdot f(x) = f(\gamma^{-1} \cdot x) \quad \forall f \in L^2_0(X, \mu), \gamma \in \Gamma.$$
The Koopman representation

**Definition**

When \( \Gamma \curvearrowright^a (X, \mu) \), the **Koopman representation**, \( \kappa_0^a \), is given by

\[
\gamma \cdot f(x) = f(\gamma^{-1} \cdot x) \quad \forall f \in L_0^2(X, \mu), \gamma \in \Gamma.
\]

**Definition**

For two actions \( \Gamma \curvearrowright^a (X, \mu) \), \( \Gamma \curvearrowright^b (Y, \nu) \), \( b \) is a **factor** of \( a \) if there is a measure preserving, surjective map \( \phi: X \rightarrow Y \) such that

\[
\phi(\gamma \cdot x) = \gamma \cdot \phi(x).
\]

\( a \) and \( b \) are **conjugate** if \( \phi \) is a measure space isomorphism.
The Koopman representation

**Definition**

When $\Gamma \bowtie^a (X, \mu)$, the Koopman representation, $\kappa^a_0$, is given by

$$\gamma \cdot f(x) = f(\gamma^{-1} \cdot x) \quad \forall f \in L^2_0(X, \mu), \gamma \in \Gamma.$$ 

**Definition**

For two actions $\Gamma \bowtie^a (X, \mu)$, $\Gamma \bowtie^b (Y, \nu)$, $b$ is a factor of $a$ if there is a measure preserving, surjective map $\phi: X \rightarrow Y$ such that

$$\phi(\gamma \cdot x) = \gamma \cdot \phi(x).$$

$a$ and $b$ are conjugate if $\phi$ is a measure space isomorphism.

If $b$ is a factor of $a$, then $\kappa^b_0 \leq \kappa^a_0$.

If $b$ is conjugate to $a$, then $\kappa^b_0 \cong \kappa^a_0$. 
Let $\{\pi_i\}_{i \in I}$ be a set of continuum many non-equivalent, irreducible weakly mixing representations of $\mathbf{F}_2$. These can be turned into actions $\mathbf{F}_2 \curvearrowright^{a_{\pi_i}} (Z_i, \nu_i)$ such that

$$\pi_i \cong \pi_j \implies \kappa_0^{a_{\pi_i}} \cong \kappa_0^{a_{\pi_j}}$$

and $\pi_i \leq \kappa_0^{a_{\pi_i}}$.

Also, take the action $\mathbf{F}_2 \curvearrowright^{a} (\mathbb{T}^2, h)$. 
Let \( \{\pi_i\}_{i \in I} \) be a set of continuum many non-equivalent, irreducible weakly mixing representations of \( F_2 \). These can be turned into actions \( F_2 \curvearrowright^{a_{\pi_i}} (Z_i, \nu_i) \) such that

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and \( \pi_i \leq \kappa_0^{a_{\pi_i}} \).

Also, take the action \( F_2 \curvearrowright^a (\mathbb{T}^2, h) \).

For each \( i \in I \), let \( c_i \) and \( d_i \) be actions of \( \Gamma \) and \( F_2 \), respectively, that are given by the lemma.
Goal: Each $c_i$ is only orbit equivalent to countably many $c_j$. 
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Suppose that there is some $i \in I$ such that there is an uncountable set $J \subset I$ such that $c_i$ is orbit equivalent to $c_j$ for all $j \in J$. 
Goal: Each $c_i$ is only orbit equivalent to countably many $c_j$.

Suppose that there is some $i \in I$ such that there is an uncountable set $J \subset I$ such that $c_i$ is orbit equivalent to $c_j$ for all $j \in J$.

(Ioana) There is an uncountable set $J_0 \subset J$ such that for all $j, k \in J_0$, there are non-null $F_2$-invariant subsets $Y_j' \subset Y_j$, $Y_k' \subset Y_k$ such that $d_j|Y_j'$ is conjugate to $d_k|Y_k'$. 
Proof

Let $i \in J_0$. Then for any $j \in J_0$,

$$
\pi_j \leq \kappa_0^{a \pi_j} \leq \kappa_0^{a \times a \pi_j}.
$$
Proof

Let $i \in J_0$. Then for any $j \in J_0$,

$$\pi_j \leq \kappa_0^{a_{\pi j}} \leq \kappa_0^{a \times a_{\pi j}}.$$

From the construction,

$$\kappa_0^{a \times a_{\pi j}} \leq \kappa_0^{d_j | Y_j'} \cong \kappa_0^{d_i | Y_i'}.$$
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Since $\kappa_0^{d_i|Y'_i} \leq \kappa_0^{d_i}$,

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Proof

Let $i \in J_0$. Then for any $j \in J_0$,

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Since $\kappa_0^{d_i|Y'_i} \leq \kappa_0^{d_i}$,

$$\pi_j \leq \kappa_0^{d_i}.$$

However, a separable representation may only contain countably many non-equivalent irreducible representations. So each $c_i$ can only be orbit equivalent to countably many $c_j$. 