Product of finite sets.

**Theorem.** If \( r \in \omega, k \in \omega, \) and \( n_i : i \in k \) are natural numbers, then there are numbers \( m_i : i \in k \) such that for every partition of the product \( \Pi_im_i \) into \( r \) many parts, one of them contains a subset of the form \( \Pi_ib_i \), where \( |b_i| \geq n_i \).
Infinite product.

Theorem. (DiPrisco, Henle) If \( r \in \omega \) and \( n_i : i \in \omega \) are natural numbers then there are natural numbers \( m_i : i \in \omega \) such that for every partition of the product \( \Pi_i m_i \) into \( r \) many Borel pieces, one of the pieces contains a subset of the form \( \Pi_i b_i \), where \( |b_i| \geq n_i \).

The proof uses a partition property of \([\omega]^{\aleph_0}\). Later, an upper bound for \( m_i \)'s has been found in terms of the Ackermann function.
Parametrized infinite product

**Theorem.** (DiPrisco, Llopis, Todorcevic) If $r \in \omega$ and $n_i : i \in \omega$ are natural numbers then there are natural numbers $m_i : i \in \omega$ such that for every partition of the product $\prod_i m_i \times \omega$ into $r$ many Borel pieces, one of the pieces contains a subset of the form $\prod_i b_i \times c$, where $|b_i| \geq n_i$ and $c \subseteq \omega$ is infinite.

The proof does not provide an upper bound for the numbers $m_i$. 
Theorem. If \( r \in \omega \) and \( n_i : i \in \omega \) are natural numbers then there are natural numbers \( m_i : i \in \omega \) and finite sets \( a_i : i \in \omega \) such that for every sequence \( \phi_i : i \in \omega \) of submeasures on \( a_i \) such that \( \phi_i(a_i) > m_i \) and every partition of the product \( \Pi_i a_i \times \omega \) into \( r \) many Borel pieces, one of the pieces contains a subset of the form \( \Pi_i b_i \times c \), where \( \phi_i(b_i) \geq n_i \) and \( c \subset \omega \) is infinite.

The proof uses creature forcing. It provides an approximately double exponential bound on the growth of the numbers \( m_i \) and sets \( a_i \).
Theorem. If $\varepsilon > 0$ is real and $n_i : i \in \omega$ are natural numbers then there are natural numbers $m_i : i \in \omega$ and finite sets $a_i : i \in \omega$ such that for every sequence $\phi_i : i \in \omega$ of measures on $a_i$ such that $\phi_i(a_i) > m_i$ and every Borel set $B \subset \prod_i a_i \times \omega \times [0,1]$ with vertical sections of Lebesgue mass at least $\varepsilon$, $B$ contains a subset of the form $\prod_i b_i \times c \times \{z\}$ where $\phi_i(b_i) > n_i$ and $c \subset \omega$ is infinite.
An application.

Suppose $P_i : i \in \omega$ are forcings obtained from $\sigma$-ideals generated by a compact collection of compact sets. (limsup infinity forcings). Then the product $\prod_i P_i$ does not add a splitting real.

This strengthens the result of Laver on product of Sacks forcing.
Creature forcing.

A setup is an atomic partial order $\mathcal{C}$ with a finite set of atoms and a norm function to the reals, preserving the order and vanishing at the atoms.

If $\mathcal{C}_i : i \in \omega$ are setups then the derived creature forcing $P$ is just the subset of $\prod_i \mathcal{C}_i$ consisting of functions on which the norms tend to infinity.

Main point. Under suitable conditions on the setups $\mathcal{C}_i$ the creature forcing is proper, bounding, and adds no splitting reals.
The creatures we need.

Let $a$ be finite, $\phi$ a submeasure on it. A creature (nonatomic element of the setup) is a pair $\langle b, r \rangle$ where $b \subset a$ and $r \geq \log(\phi(b) + 1)$. Define

- $\text{norm}(b, r) = \varepsilon \log(\log(\phi(b) + 1) - r + 1)$

- $\langle c, s \rangle \leq \langle b, r \rangle$ if $c \subset b$ and $s \geq r$. 