Forcing With Ultrafilters and Forcing Ultrafilters

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Definitions and Conventions

Ultrafilters are non-principal (unless the contrary is explicitly stated) and on ω (or other countable sets).

An ultrafilter $\mathcal{U}$ is *selective* if every function on ω is one-to-one or constant on some set in $\mathcal{U}$.

An ultrafilter $\mathcal{U}$ is a *P-point* if every function on ω is finite-to-one or constant on some set in $\mathcal{U}$.

An ultrafilter $\mathcal{U}$ is *rapid* if every function $\omega \to \omega$ is majorized by the increasing enumeration of some set in $\mathcal{U}$.

If $f : \omega \to \omega$ and $\mathcal{U}$ is an ultrafilter, then

$$f(\mathcal{U}) := \{X \subseteq \omega : f^{-1}(X) \in \mathcal{U}\}$$

is a (possibly principal) ultrafilter.

$\mathcal{V} \leq \mathcal{U}$ means that $\mathcal{V} = f(\mathcal{U})$ for some f. This is the *Rudin-Keisler* ordering, often written $\leq_{RK}$.

The Ideal Situation (Mathias)

The following are equivalent notions of forcing:

- infinite subsets of ω ordered by $\subseteq^*$,
- $(\mathcal{P}(\omega)/\text{fin}) - \{0\}$,
- countably generated filters on ω, ordered by $\supseteq$.

The generic object is a selective ultrafilter.

Selective ultrafilters have partition properties.

- $\omega \to [\mathcal{U}]^n_k$ for all finite n and k. (Kunen)
- $\omega \to [\mathcal{U}]^\omega_2$ for partitions into an analytic piece and a coanalytic piece. (Mathias)
- In the "Lévy-Mahlo" model gotten by Lévy-collapsing to ω all cardinals below a Mahlo cardinal κ, $\omega \to [\mathcal{U}]^\omega_2$ for $OD(\mathbb{R})$ partitions. (Mathias)

Corollary: In the Lévy-Mahlo model, every selective ultrafilter is generic over $HOD(\mathbb{R})$ with respect to $[\omega]^\omega$-forcing.

One says that selectivity is *complete combinatorics* for this forcing.

The Ideal Situation, continued

The forcing $[\omega]^\omega$ provides a base for the usual topology of the Stone-Čech remainder $\omega^* = \beta\omega - \omega$.

Because it's countably closed, there's a stronger Baire category theorem: Every intersection of $\aleph_1$ dense open sets is dense.

The selective ultrafilters are the intersection of $\mathfrak{c} = 2^{\aleph_0}$ dense open sets.

So CH gives "lots" of selective ultrafilters.

In the absence of CH, there might be none, e.g., in the random real model. (Kunen)

Mathias forcing with respect to an ultrafilter $\mathcal{U}$ has conditions $\langle s, A\rangle$, where s is a finite subset of ω and $A \in \mathcal{U}$. The condition "says" of the generic set $M \in [\omega]^\omega$ that it includes s and that $M - s \subseteq A$. Conditions are ordered accordingly.

The Mathias-generic M is almost included in each set from $\mathcal{U}$. If $\mathcal{U}$ is selective, or merely rapid, then the increasing enumeration of M eventually dominates every real from the ground model.

The F_σ Situation

Another, finer topology on ω^* has as a basis of open sets the closed (in the usual topology) sets determined by F_σ filters $\mathcal{F}$ on ω, i.e., the sets

$$\{\mathcal{U} \in \omega^* : \mathcal{F} \subseteq \mathcal{U}\}$$

where $\mathcal{F}$ is F_σ as a subset of $\mathcal{P}(\omega) \cong 2^\omega$.

Again there is a stronger Baire category theorem. Daguenet showed that, under CH, one has a "comeager" (in this sense) set of ultrafilters that are P-points but not selective, not even rapid, and not even $\geq$ any rapid ultrafilters.

The same proof shows that forcing with F_σ filters produces a generic ultrafilter that is a P-point $\geq$ no rapid ultrafilter.

Non-Dominating Mathias Forcing

Canjar studied ultrafilters whose Mathias forcing adds no dominating real.

He showed, under CH, that such ultrafilters exist.

He also showed that they must be P-points $\geq$ no rapid ultrafilter.

Strong P-Points

Laflamme showed that F_σ-generic ultrafilters $\mathcal{U}$ are *strong P-points.*

This means that, if $\mathcal{C}_n$ for $n \in \omega$ are closed (in 2^ω) subsets of $\mathcal{U}$, then there is a partition of ω into finite intervals $I_n = [i_n, i_{n+1})$ (with $0 = i_0 < i_1 < \dots$) such that, for any sets $X_n \in \mathcal{C}_n$,

$$\bigcup_{n \in \omega} (X_n \cap I_n) \in \mathcal{U}.$$

This implies that $\mathcal{U}$ is a P-point $\geq$ no rapid ultrafilter.

Laflamme also showed that, if Mathias forcing with respect to $\mathcal{U}$ adds no dominating reals, then $\mathcal{U}$ is a strong P-point.

He also showed that any ultrafilter generated by $< \mathfrak{d}$ closed subsets is a strong P-point.

Known earlier: An ultrafilter generated by $< \mathfrak{d}$ elements is a P-point (Ketonen) with no rapid ultrafilter below it (trivial).

Non-Dominating Mathias Forcing

Hrušák and Minami have recently announced the following combinatorial condition on ultrafilters $\mathcal{U}$ that is necessary and sufficient for $\mathcal{U}$-Mathias forcing to not add dominating reals.

First, define $\mathcal{U}^{<\omega}$ to be the filter, on the family $\mathbb{F}$ of finite nonempty subsets of ω, generated by the sets

$$A^{<\omega} = \{a \in \mathbb{F} : a \subseteq A\}$$

for $A \in \mathcal{U}$.

The *Hrušák-Minami* (HM) condition (on $\mathcal{U}$) is the assertion that $\mathcal{U}^{<\omega}$ is a P^+-filter. This means that every decreasing sequence of sets of positive measure (with respect to $\mathcal{U}^{<\omega}$) has a pseudo-intersection of positive measure.

The Hrušák-Minami theorem is that $\mathcal{U}$ satisfies the HM condition if and only if Mathias forcing with respect to $\mathcal{U}$ adds no dominating reals.

Back to Strong P-points

Theorem: An ultrafilter satisfies the HM condition if and only if it is a strong P-point.

Conjecture: In the Lévy-Mahlo model, every strong P-point is generic over $HOD(\mathbb{R})$ with respect to F_σ filter forcing.

In other words, "strong P-point" or the equivalent HM condition is conjectured to be complete combinatorics for F_σ filter forcing.