$\mathcal{G}_0$-dichotomies for ∞-Borel sets

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Introduction

The topic of this talk was motivated by Benjamin Miller's recent results.

One of the key aspects of Miller's work is the use of "classical" arguments. By contrast, the results here use forcing, large cardinals, and consequences of determinacy.

The other key aspect, that we definitely take advantage of, is the *soft* deduction of many dichotomy theorems in classical descriptive set theory from graph theoretic dichotomies via Baire category arguments.

As shown through Miller's talks, these graph theoretic dichotomies can be established in a vast generality, depending only in the existence of Suslin representations for the relevant sets. Thus, they hold under $\mathsf{AD}_{\mathbb{R}}$ of arbitrary graphs on $\mathbb{R}$.

Ketchersid and I have shown that the appropriate versions of these graph theoretic dichotomies hold, for example, in natural models of AD^+, thus obtaining by soft arguments the other dichotomies as well.
Since in general models of AD^+ not all sets of reals are Suslin, and the dichotomies do not seem to reduce to the Suslin case by the usual reflection arguments, an approach different from Miller's is needed; we use arguments involving Vopěnka-like forcing.
I want to concentrate on the proof of the $\mathcal{G}_0$-dichotomy of Kechris-Solecki-Todorčević in this context, to illustrate the technique.

Graphs and colorings

As in Miller's framework, for us a graph G on a set X is a digraph, i.e., a subset of X^2.
Given such G, a Y-coloring of G is a function $c : X \to Y$ such that

$$G(x_0, x_1) \implies c(x_0) \neq c(x_1).$$

A set $A \subseteq X$ is G-discrete, or *independent* iff $A^2 \cap G = \emptyset$, so $c : X \to Y$ is a Y-coloring of G iff for all $y \in Y$, $c^{-1}[\{y\}]$ is G-discrete.
We will be interested in coloring with ordinals.

Fix $\mathbf{s} = (s_n \mid n \in \omega)$ dense in $2^{<\omega}$ with $s_n \in 2^n$.

Define the graph $\mathcal{G}_0$ on 2^ω by:

$$\mathcal{G}_0(x_0, x_1) \iff \exists n \exists x \forall i < 2\, (x_i = s_n{}^\frown(i)^\frown x).$$

Restriction on colorings of $\mathcal{G}_0$.

An immediate but key fact about $\mathcal{G}_0$ mentioned in Miller's talks:

Fact

Any $\mathcal{G}_0$-discrete set A with the property of Baire must be meager.

The proof is a straightforward Baire category argument.
Thus for any Baire-measurable coloring $c : 2^\omega \to Y$ of $\mathcal{G}_0$, $c^{-1}[\{y\}]$ is meager, hence, meager sets can not be closed under $|Y|$-sized unions.
This places limitations on definable colorings. For example, there can not be a Baire measurable ω-coloring, or (under AD) any colorings by ordinals.

$\mathcal{G}_0$-dichotomy for analytic graphs:

For G an analytic graph on $\mathbb{R}$ exactly one of the following hold:

1. G is ω-colorable via a Borel measurable map.
2. There is a continuous map $\pi : 2^\omega \to \mathbb{R}$ so that π is a homomorphism of $\mathcal{G}_0$ into G.

The second possibility will be denoted $\mathcal{G}_0 \leq_c G$.
(We already knew that these possibilities are mutually exclusive.)

To state our results, it is best to work in a fragment of AD^+. We need the notion of ∞-Borel sets. Essentially, we generalize the iterative definition of the Borel hierarchy, by allowing well-orderable unions and intersections (of any length).

Since we work without choice, rather than the sets themselves, we are more interested in their actual construction. Define the class $\mathrm{bc}_{<\kappa}$ of $<\kappa$-Borel codes, for κ a cardinal, as the collection of well-founded trees on $\gamma<\kappa$ describing how to build a set of reals by taking well-ordered unions and complements from basic sets.

More precisely, think of reals as subsets of ω. A code S can be seen as a formula ϕ_S in the propositional language $\mathcal{L}_{\infty,0}$, where we allow the use of countably many propositional variables p_i.
The code S describes the set $\{x \in \mathbb{R} \mid x \models \phi_S\}$, where the semantics are defined in the standard way, after setting $x \models p_i$ iff $i \in x$.
A $<\kappa$-code is then a tree, and can be identified with a set of ordinals bounded below κ. An ∞-Borel code is a $<\kappa$-Borel code for some κ.

Given a $< \kappa$-Borel code S, write $S(x)$ to mean "x is in the set coded by S." This is very absolute:

$$S(x) \iff L_{o(S,x)}[S,x] \models \phi(S,x),$$

where $o(S,x) = \omega_1^{CK}(S,x)$ is the first admissible over S, x and ϕ is an appropriate Σ_1-formula.
If it is important to distinguish S from the set it codes, we write $A_S, B_S, \dots$ for the latter.
A $< \kappa$-Borel set is the interpretation of a $< \kappa$-Borel code. Denote by $\mathbb{B}_{<\kappa}$ the class of $< \kappa$-Borel sets. An ∞-Borel set is a $< \kappa$-Borel set for some κ.

Suslin sets have strong absoluteness properties. In an attempt to generalize regularity results that hold in the Solovay model or under AD^+ about κ-Suslin sets, we weaken the assumption of being Suslin to simply carrying an ∞-Borel code, and use Łoś's lemma on ultrapowers to replace the use of absoluteness. For this, it is convenient to work in the following theory:

Definition (BT)

- $\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}}$.
- There is a fine σ-complete measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$.

BT holds, for example, in the Solovay model after Levy collapsing a measurable cardinal to ω_1 and in models of Turing-determinacy, assuming $\mathsf{DC}_{\mathbb{R}}$.

Our results apply to models of Woodin's AD$^+$:

Definition (AD$^+$)

- $\mathsf{DC}_{\mathbb{R}}$.
- $< \Theta$-ordinal determinacy.
- All sets of reals are ∞-Borel.

If AD$^+$ holds in a model M, then it holds in $L(\mathcal{P}(\mathbb{R}))^M$. We say that a *natural model* of AD$^+$ is one satisfying $V = L(\mathcal{P}(\mathbb{R}))$. It is on these models that we concentrate.

$\mathcal{G}_0$-dichotomy

Assuming BT:

$\mathcal{G}_0$-dichotomy for ∞-Borel graphs (C-Ketchersid)

Let μ be a fine σ-complete measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$. Suppose G is a $< \kappa$-Borel graph with code S. Then exactly one of the following holds:

1. There is a $\mathbb{B}_{<\kappa_S^\infty}$-measurable κ_S^∞-coloring.
2. $\mathcal{G}_0 \leq_c G$.

Here $\kappa_S^\infty = \prod_\tau \kappa_S^\tau / \mu$ where κ_S^τ is the first inaccessible of $\mathrm{HOD}_S^{L(S,\tau)}$.

The extent of the ∞-Borel sets

Under the assumption of BT, Woodin saw how to associate a code $\exists \dot{\boldsymbol{y}} S$ for a subset of $\mathbb{R}^m$ to a code S for a subset of $\mathbb{R}^{n+m}$ such that

$$(\exists \dot{\boldsymbol{y}} S)(\boldsymbol{x}) \iff \exists \boldsymbol{y} S(\boldsymbol{x}, \boldsymbol{y})$$

for all $\boldsymbol{x} \in \mathbb{R}^m$. This easily yields that if $S \subseteq \mathrm{ORD}$, then every set of reals in $L(S, \mathbb{R})$ is ∞-Borel. From this we can easily prove (by adapting Solovay's argument) that all sets of reals in $L(S, \mathbb{R})$ have the standard regularity properties, in particular, the Baire property. It follows that all functions $f : \mathbb{R} \to \mathrm{ORD}$ are Baire measurable. (For our result, we actually need an explicit computation that allows us to bound the size of $\exists \dot{\boldsymbol{y}} S$ in terms of the size of S.)

Given sets $S_1, \ldots, S_k$, we denote by $\mathsf{HOD}_{S_1,\ldots,S_k}$ the collection of hereditarily ordinal definable sets with parameters from $\mathsf{ORD} \cup \{S_1, \ldots, S_k\}$.
For $\sigma \in \mathcal{P}_{\omega_1}(\mathbb{R})$ and $S \subset \mathsf{ORD}$, let $H^\sigma_S = (\mathsf{HOD}_S)^{L(S,\sigma)}$.
Denote by bc^σ_S the class $(\mathsf{bc})^{H^\sigma_S}$ of Borel codes in H^σ_S, and define $T \sim^\sigma_S T'$ for $T, T' \in \mathsf{bc}^\sigma_S$ iff $(A_T = A_{T'})^{L(S,\sigma)}$.
Let $\mathbb{Q}^\sigma_S$ be the Vopěnka algebra $\mathsf{bc}^\sigma_S / \sim^\sigma_S$. Note that $\sim^\sigma_S$ is ordinal definable in $L(S,\sigma)$ from S. The poset $\mathbb{Q}^\sigma_{S,<\kappa}$ is defined similarly, restricting to the set $\mathsf{bc}^\sigma_{S,<\kappa} = (\mathsf{bc}_{<\kappa})^{H^\sigma_S}$.

By κ_S^σ we denote the first "true inaccessible" of $L(S,\sigma)$. What we want is to use κ_S^σ as a bound for the size of the antichains in $\mathbb{Q}_S^\sigma$. It is easy to see that it suffices to take κ_S^σ to be the least regular κ such that for every $\alpha < \kappa$ there is no surjection from $\mathcal{P}(\mathbb{R} \times \alpha)$ onto κ. It is straightforward to verify that κ_S^σ coincides with the first strongly inaccessible cardinal in $L(S,\sigma)^{\mathtt{Coll}(\mathbb{R},\omega)}$, which is a model of choice.

Fix a fine σ-complete measure μ on $\mathcal{P}_{\omega_1}(\mathbb{R})$. Note that using μ we can easily define a (normal) measure μ_{ω_1} on ω_1. It is easy to check that

$$\mathsf{HOD}_{S,x,\mu_{\omega_1}} \models \text{“}\omega_1^V \text{ is measurable”}$$

for any real x. It follows that ω_1^V is Mahlo in $L[S,x]$ and therefore in $L(S,\sigma)$ whenever $\sigma \in \mathcal{P}_{\omega_1}(\mathbb{R})$ and x is a real coding the range of σ. In particular, $\mathbb{R} \cap L(S,\sigma)$ is countable and, moreover, κ_S^σ is well-defined, and its power set in $L(S,\sigma)$ is countable in V.

It turns out that for μ-a.e σ, $\mathbb{Q}^\sigma_S = \mathbb{Q}^\sigma_{S,<\kappa^\sigma_S}$. This allows us to bound the complexity of the colorings we obtain.
Our arguments use ultrapowers. Here, $\prod_S A^\sigma/\mu$ is the version of the ultrapower of the sets A^σ, $\sigma \in \mathcal{P}_{\omega_1}(\mathbb{R})$, where only S-invariant functions are used. A function $f : \mathcal{P}_{\omega_1}(\mathbb{R}) \to V$ is S-invariant iff $f(\sigma) = f(\tau)$ whenever $L(S,\sigma) = L(S,\tau)$.

The following observations are key features of $\mathbb{Q}^\sigma_S$:

- For all $x \in \sigma$, $G_x = \{T \in \mathbb{Q}^\sigma_S : T(x)\}$ is H^σ_S-generic.
- $H^\sigma_S[x] = H^\sigma_S[G_x]$.
- For every $T \in \mathbb{Q}^\sigma_S$, there is $x \in \sigma$ with $T \in G_x$.

The equality

$$\mathbb{Q}_S^\sigma = \mathsf{bc}_\infty^{H_S^\sigma} / \sim_S^\sigma = \mathsf{bc}_{<\kappa_S^\sigma}^{H_S^\sigma} / \sim_S^\sigma$$

gives us that $\mathbb{Q}_S^\sigma$ is κ_S^σ-cc, so the forcing and all of its maximal antichains are contained in $H_S^\sigma \cap V_{\kappa_S^\sigma}$.
Define $\langle H_S^\infty, \mathbb{Q}_S^\infty, \kappa_S^\infty \rangle = \prod_S \langle H_S^\sigma, \mathbb{Q}_S^\sigma, \kappa_S^\sigma \rangle / \mu$.
By Łoś's lemma we have:

- For all $x \in \mathbb{R}$, $G_x = \{T \in \mathbb{Q}_S^\infty : T(x)\}$ is H_S^∞-generic.
- $H_S^\infty[x] = H_S^\infty[G_x]$.
- $T \sim_S^\infty T' \iff \forall x \in \mathbb{R}[T(x) \iff T'(x)]$, so $\mathbb{Q}_S^\infty = \mathsf{bc}_{<\kappa_S^\infty}^{H_S^\infty} / \sim_S^\infty$, and the corresponding interpretations are in $\mathbb{B}_{<\kappa_S^\infty}$.
- For every $T \in \mathbb{Q}_S^\infty$, there is $x \in \mathbb{R}$ with $T \in G_x$.

The proof

Assume BT. We want to show a quantitative version of the fact that if G is an ∞-Borel graph on $\mathbb{R}$, then either there is a coloring $f : \mathbb{R} \to$ ORD of G, or else there is a continuous homomorphism of $\mathcal{G}_0$ to G.

Let μ be a fine σ-complete measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$. We will work primarily with $\mathbb{Q}_S^\infty$ and so omit the superscript; when S is clear from context we omit it as well.

By $\mathbb{Q}^n$ we denote the version of $\mathbb{Q}$ for subsets of $\mathbb{R}^n$. This is not the same as $\underbrace{\mathbb{Q} \times \cdots \times \mathbb{Q}}_{n \text{ times}}$.

For $p \in \mathbb{Q}^n$, denote by p^2 the code in $\mathbb{Q}^{2n}$ such that

$$p^2(\dot{x}_0, \ldots, \dot{x}_{2n-1}) \iff p(\dot{x}_0, \ldots, \dot{x}_{n-1}) \wedge p(\dot{x}_n, \ldots, \dot{x}_{2n-1}).$$

Fix an ∞-Borel code S for the graph G. Let

$$W^\infty = \{p \in \mathbb{Q} : H^\infty \models p^2 \Vdash_{\mathbb{Q}^2} \neg G_{S^\infty}(x_0, x_1)\},$$

where (x_0, x_1) is the standard name for the $\mathbb{Q}^2$-generic pair. If this set is dense, we are done, since there is a natural well-ordering of the elements of $\mathbb{Q}$, every pair of reals (in V) is $\mathbb{Q}^2$-generic over H^∞, and the map that assigns to each real r the least condition p in the generic corresponding to r, is clearly a coloring of G; the point is that $G_{S^\infty} \cap (\mathbb{R}^2)^V = G$, hence if $\neg G_{S^\infty}(y_0, y_1)$ then, in fact, $\neg G(y_0, y_1)$.

We are left with the task of building the homomorphism of $\mathcal{G}_0$ to G when W^∞ fails to be dense. Pick a witness p, so for any $p' \leq p$ there is $c \in \mathbb{Q}^2$ with $c \leq {p'}^2$ and such that

$$H^\infty \models c \Vdash G_{S^\infty}(x_0, x_1).$$

The construction is inductive. For $n \in \omega$ let $A_n =$

$$\{(t_0, t_1) \in (2^n)^2 : \exists m < n \exists u \in 2^{<\omega} \forall i < 2 (t_i = s_m {}^\frown (i) {}^\frown u)\}.$$

Note that $A_0 = \emptyset$ and

$$A_{n+1} = \{(t_0 i, t_1 i) : (t_0, t_1) \in A_n, i \in 2\} \cup \{(s_n 0, s_n 1)\}.$$

(In Miller's notation, $A_n = \mathcal{G}_0(2^n)$.)

We build a Lipschitz map from 2^ω into 2^ω, and use a forcing argument to guarantee that it works as the required homomorphism. For this, we identify a σ so that the map can be in fact seen as an assignment of $\mathbb{Q}^\sigma$-generics over H^σ to reals, and for this we build some $\mathbb{Q}^\sigma$-conditions (ensuring genericity) along the way.

The key fact is the following preliminary lemma:

Lemma

Suppose that $q \in \mathbb{Q}^{2^n}$ is such that q is below the 2^n-fold sum p^{2^n}. For any $s \in 2^n$, we have $q^2 \wedge [\![G_{S^\infty}(x_{s0}, x_{s1})]\!]_{\mathbb{Q}^{2^{n+1}}} > 0$.

Using Łoś's lemma, fix a countable σ for which the corresponding version of the key lemma holds. By fixing an *appropriate* sequence $(D_n : n < \omega)$ so that D_n is dense in $\mathbb{Q}^{n,\sigma}$, we can ensure the following:
Suppose that a sequence of conditions

$$(q_n : n < \omega)$$

is such that $q_n \in D_n$ and $q_{n+1} \leq q_n^2 \in \mathbb{Q}^{2^{n+1}}$. Then this sequence generates $\mathbb{Q}^{m,\sigma}$-generics over H^σ in a natural way: Consider any pairwise different $s_0, \ldots, s_m \in 2^\omega$, $m < \omega$. Given n, let

$$\{s_0, \ldots, s_m\} \restriction n = \{s_0 \restriction n, \ldots, s_m \restriction n\}.$$

Let k be such that $\{s_0, \ldots, s_m\} \restriction k$ has size m. Then the sequence of conditions

$$\{\pi_n(q_n) : n \geq k\}$$

is $\mathbb{Q}^{m,\sigma}$-generic over H^σ, where π_n is an appropriate projection on the set of coordinates $s_i \restriction n$ for $i \leq m$.

Assume we have defined an approximation $\sigma_n : 2^n \to 2^n$, and we want to build σ_{n+1}. We have also identified some condition $q_n \in \mathbb{Q}^{2^n,\sigma}$ with the property that the appropriate projections meet the first n sets D_i, and if $(\dot{x}_s)_{s\in 2^n}$ is the standard name for the generic real, then q_n decides each $\dot{x}_s \restriction n$, and $\sigma_n(s)$ is precisely this sequence. Moreover, if $(t_0, t_1) \in A_n$ then $q_n \Vdash G_S\ (x_{t_0}, x_{t_1})$. To extend, we find $q_{n+1} \in \mathbb{Q}^{2^{n+1},\sigma}$ below q_n^2 guaranteeing the above for $n+1$ instead of n, and the map σ_{n+1} is given by $\sigma_{n+1}(s) = t$ iff

$$q_{n+1} \Vdash \dot{x}_s \restriction n+1 = t$$

for $s, t \in 2^{n+1}$.

To do this extension, simply note that the "σ-version" of the key lemma applies to q_n. Extend $q_n^2 \wedge [\![G_S\ (x_{s_n0}, x_{s_n1})]\!]_{\mathbb{Q}^{2^{n+1}},\sigma}$ so it meets D_{n+1} and decides each $\dot{x}_s \upharpoonright n+1$ for $s \in 2^{n+1}$.
As observed above, A_{n+1} is obtained by taking 2 copies of A_n and adding the tuple (s_n0, s_n1). Of this last tuple we took care explicitly through the key lemma. The other tuples are already taken care of, since the corresponding reals are in one of the 2 "copies" of q_n.
This completes the inductive construction of the Lipschitz map $\Sigma = \bigcup_n \sigma_n$.

Finally, we argue that Σ is indeed a homomorphism of $\mathcal{G}_0$ to G.
For if $\mathcal{G}_0(r_0, r_1)$, then there is some m and some $t \in 2^\omega$ such that $r_i = s_m{}^\frown(i)^\frown t$ for all $i \in 2$.
Carrying out the construction above, we have that $(\Sigma(r_0), \Sigma(r_1))$ is $\mathbb{Q}^{2,\sigma}$-generic over H^σ, and (in particular) $G_S(\Sigma(r_0), \Sigma(r_1))$, by genericity.
This completes the proof.

Applications

We obtain appropriate versions of many other dichotomy theorems from the $\mathcal{G}_0$-dichotomy and its extensions. In particular, we have:

Theorem (C-Ketchersid)

Under BT, $\mathbb{R}/E_0$ *is a successor of* $\mathbb{R}$.

This was previously known under $\mathrm{AD}_{\mathbb{R}}$ but not in general (for example, in $L(\mathbb{R})$ under AD).

We can also establish appropriate versions of the Glimm-Effros dichotomy, and recover the following trichotomy result, that we had established in earlier work:

Theorem (C-Ketchersid)

In models of BT *of the form* $V = L(S, \mathbb{R})$, *or in natural models of* AD^+, *for* **every** *set* X, *exactly one of the following holds:*

1. X *is well-orderable.*
2. X *is linearly orderable, but not well-orderable. In this case,* $|\mathbb{R}| = |2^\omega| \leq X \leq |2^\kappa|$ *for some (well-ordered)* κ.
3. $|\mathbb{R}/E_0| \leq |X|$.

These results are the first steps towards understanding "small" cardinalities of not necessarily well-orderable sets in natural models of AD^+.

- There are no infinite Dedekind-finite sets.
- Any infinite well-ordered κ has exactly two successors: κ and $\kappa + \mathbb{R}$.
- $\mathbb{R}$ has at least two successors: $\mathbb{R} + \omega_1$ and $\mathbb{R}/E_0$. There are sets larger than $\mathbb{R}$ that do not embed either of these. They are all linearly orderable, but do not embed ω_1. Some of them are successors, but at the moment there is no full classification.

The end.