Set theory

Why and which forcing axioms?

How to obtain a forcing axiom

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Using SUSIFA

Future work

SUSIFA, a new kind of forcing axiom

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University of East Anglia

Luminy set theory workshop, October 5, 2010

Why and which forcing axioms?

Discussion.

More forcing axioms?

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SCH If κ is such that $2^{\mathrm{cf}(\kappa)} < \kappa$, then $\kappa^{\mathrm{cf}(\kappa)} = \kappa^+$.

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So SCH $\implies$ there cannot be a singular strong limit κ with $2^\kappa > \kappa^+$.

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Theorem

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This explains that even simplest forcing is complex at a singular cardinal (and consequently at its successor) is not trivial.

How to make a forcing axiom

Discussion on the board.

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the universality number of the family of graphs on κ^+ is κ^{++} (so $<< 2^{\kappa^+}$).

$\mathrm{SUSIFA}_{\theta,\partial,\lambda}(\kappa, u)$ stands for:

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(4) $\mathcal{P}(\kappa^+) \subseteq \bigcup\{\mathbf{V}_i : i < \partial\}$ and

there are disjoint subsets S_1 and S_2 of ∂ such that S_1 is unbounded, S_2 is a club and ...

(a) for $i \in S_1$, $\mathbf{V}_{i+1}$ satisfies $\mathrm{SFA}(\kappa)$ and $\mathbf{V}_{i+1} \models$ "$2^\kappa > |\mathcal{P}(\kappa^+)|^{\mathbf{V}_i}$",

(b) for each $i \in S_2$, $u_i = u \restriction \mathbf{V}_i \in (U_\infty)^{\mathbf{V}_i}$,

(c) there is a sequence $\bar{A} = \langle A_i : i \in S_2 \rangle$ such that each $A_i \in \mathfrak{F}(u)$ and for all $B \in \mathfrak{F}(u_i)$, $|B \setminus A_i| < \kappa$, and $A_i \in \mathbf{V}_{i+1}$.

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Picture on the board.

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$R(u)$ names over $\mathbf{V}$ can be seen as a sequence of $\langle R(u_i) : i \in S_2\rangle$ names

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Graph G on κ^+–>

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pregraph: $M = (\kappa^+, R_\alpha)_{\alpha<\kappa}$ where R_αs are subsets of the set of unordered pairs from κ^+.

The intuition is that a pregraph gives us κ many choices for a graph and the generic choses one.

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Definition

Let pr in **V** be any bijection $\mathrm{pr} : \kappa \times \kappa \to \kappa$.

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We define a $\mathbb{R}(u)$-name $\dot\gamma(\Gamma) = (\kappa^+, \dot R)$ for a graph on κ^+ by letting $\zeta \dot R \xi$ for $\zeta < \xi < \kappa^+$ iff $R_{\mathrm{pr}(\alpha,\beta)}(\zeta, \xi)$ holds for some β and some $p \in \dot G$ forces that "$\dot h(\alpha) = 1$".

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h is a certain sequence of κ many names for a truth value.

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Lemma

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(2) Suppose that Γ and Γ' are two κ-coloured pregraphs on κ^+ and $g : \kappa^+ \to \kappa^+$ is such that g induces an embedding from Γ to Γ'. Then it is forced that "g is an embedding from $\dot{\gamma}(\Gamma)$ to $\dot{\gamma}(\Gamma')$".

Conclusion

Suppose that $\{\Gamma_i : i < \partial\}$ *is a universal subfamily for the family of all* κ*-coloured* κ^+*-pregraphs in* $\mathbf{V}$.

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Suppose that $\{\Gamma_i : i < \partial\}$ *is a universal subfamily for the family of all* κ*-coloured* κ^+*-pregraphs in* $\mathbf{V}$.

Then in the extension $\mathbf{V}[G]$ *by* $\mathbb{R}(u)$*, the family* $\{(\dot{\gamma}(\Gamma_i))_G : i < \partial\}$ *is a universal family for the family of all graphs on* κ^+.

Now we need to make a universal subfamily for the family of all κ-coloured κ^+-pregraphs in **V**.

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We make sure that if $i \in S_1$, then there is a κ-coloures κ^+-pregraph in $\mathbf{V}_{i+1}$ which embeds all κ-coloured κ^+-pregraphs in $\mathbf{V}_i$.

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We shall use the axiom SFA. Work now in $\mathbf{V}_{i+1}$.

Let $\Upsilon = |\mathcal{P}(\kappa^+)|^{V_i}$, so $\Upsilon < 2^\kappa$.

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NOTE: Since κ is supercompact in $\mathbf{V}$, $\kappa^{<\kappa} = \kappa$, and therefore $\kappa^{<\kappa} = \kappa$ also holds in $\mathbf{V}_i$.

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NOTE: Since κ is supercompact in $\mathbf{V}$, $\kappa^{<\kappa} = \kappa$, and therefore $\kappa^{<\kappa} = \kappa$ also holds in $\mathbf{V}_i$.

In particular $\zeta < \kappa^+ \implies |\{\eta_\alpha \restriction \zeta : \alpha < \Upsilon\}| \leq \kappa^+$.

Let $\langle\langle M_\alpha, (R_{\alpha,\varepsilon})_{\varepsilon<\kappa}\rangle : \alpha < \Upsilon\rangle$ be a list without repetitions of all κ-coloured pregraphs on κ^+ from $\mathbf{V}_i$.

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Definition

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where:

(i) $\iota^p \in [\Upsilon]^{<\kappa}$, $v^p \in [\kappa]^{<\kappa}$,
(ii) for $\alpha \in \iota^p$, we have that f^p_α is a partial one-to-one function from κ^+ with $|\mathrm{dom}(f^p_\alpha)| < \kappa$, mapping $\zeta \in \mathrm{dom}(f^p_\alpha)$ to an element of $\{\eta_\alpha \restriction \zeta\} \times \kappa$,

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(iii) for $\varepsilon \in v^p$, $R^{*,p}_\varepsilon$ is a subset of the set of unordered pairs from $\bigcup_{\alpha \in \iota^p} \mathrm{rge}(f^p_\alpha)$,
(iv) for every $\alpha \in \iota^p$ and $\zeta < \xi \in \mathrm{dom}(f^p_\alpha)$, for every $\varepsilon < \kappa$, $R^{*,p}_\varepsilon(f^p_\alpha(\zeta), f^p_\alpha(\xi))$ iff $R_{\alpha,\varepsilon}(\zeta, \xi)$.

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This obviously does the job,

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This obviously does the job, the point is that SFA applies to this forcing:

Definition

Axiom $\mathrm{SFA}(\kappa)$:

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P

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Axiom $\mathrm{SFA}(\kappa)$: for every $(<\kappa)$-directed closed forcing *P* which has $(2,\omega)$-lubs, if *P* is stationary κ^+-cc, then for every family $\mathfrak{D}$ of $< 2^\kappa$ dense sets in *P*, there is a filter in *P* which intersects every set in $\mathfrak{D}$.

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RC (Todorčević reformulation) A tree is special iff every of its subsets of size $\aleph_1$ is special.

Reflecton?

At successor of regular cardinals, reflection principles and forcing axioms contradict each other:

e.g. MA contradicts the Rado conjecture.

RC (Todorčević reformulation) A tree is special iff every of its subsets of size $\aleph_1$ is special.

With Victor Torres we shall investigate the possibility of SUSIFA holding with reflection, such as the analogues of Rado conjecture.

The work here cannot be trivial, because of the fact that at the successor of a strong limit singular, a (rather weak) version of square must hold (Dž.-Shelah 1995).