Shelah Classification and Higher Descriptive Set Theory

Shelah's Classification Theory

T countable, complete, first-order

T is *classifiable* iff there is a "structure theory" for its models
Example: Algebraically closed fields (transcendence degree)

T is *unclassifiable* otherwise
Example: Dense linear orderings

Shelah's Characterisation (Main Gap): T is classifiable iff T is superstable without the OTOP and without the DOP

A classifiable T is *deep* iff it has the maximum number of models in all uncountable powers (Example: Acyclic undirected graphs, every node has infinitely many neighbours)

Shelah Classification and Higher Descriptive Set Theory

Another way of classifying theories: Descriptive Set Theory

Mod_T^ω = Models of T with universe ω

Isom_T^ω = The Equivalence Relation of Isomorphism on Mod_T

Isom_T^ω is an analytic (boldface Σ_1^1) equivalence relation

Classify T according to the complexity of Isom_T^ω:

Countably many classes
Smooth
Essentially countable
Borel
S_∞ complete (bireducible with Graph Isomorphism)

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Bad news: The complexity of Isom_T^ω is not a good measure of the model-theoretic complexity of T:

Dense Linear Order is bad model-theoretically but Isom_T^ω is trivial

(Koerwien) There are very classifiable theories T such that Isom_T^ω is not even Borel

Theme of this lecture: Instead use Isom_T^κ for an uncountable κ (joint work with Tapani Hyttinen and Vadim Kulikov)

Preview: For appropriate κ

T is classifiable and shallow (i.e. not deep) iff Isom_T^κ is "Borel"

T is classifiable iff for all regular $\lambda < \kappa$, Isom_T^κ is not "Borel above" equality modulo the λ-nonstationary ideal

Higher Descriptive Set Theory: Generalised Baire Space

First we have to understand what is meant by “Borel” and “Borel reducible” in the generalised Baire space κ^κ

Fix an uncountable κ such that $\kappa^{<\kappa} = \kappa$

Then Baire space ω^ω generalises nicely to κ-Baire space κ^κ:

Points in κ^κ are functions $f : \kappa \to \kappa$
Basic open sets are of the form $N_p = \{f \mid p \subseteq f\}$, $p \in \kappa^{<\kappa}$
Basic open sets are also closed
There are only κ many basic open sets
The intersection of $< \kappa$ open sets is open

Higher Descriptive Set Theory: Borel Sets

Borel sets: Close the basic open sets under unions of size κ and complements

Now we start to see a difference for uncountable κ:

Borel is a proper subclass of Δ^1_1

This is because Borel sets are described by well-founded trees and well-foundedness is Δ^1_1 for regular uncountable kappa

Higher Descriptive Set Theory: Regularity Properties

Classical DST: LM (Lebesgue Measurability), BP (Baire Property) and PSP (Perfect Set Property)

Higher DST: BP and PSP

Baire Property

The Baire Category Theorem works: The intersection of κ-many open dense sets is dense

X is *nowhere dense* iff it is contained in a closed set with no interior
Meager $=$ Union of κ-many nowhere dense sets

X has the *Baire property (BP)* iff its symmetric difference from some open set is meager

Fact: Borel sets have the BP

Higher Descriptive Set Theory: Regularity Properties

Surprise! There are Σ^1_1 sets without the BP:

Theorem

(Halko-Shelah) For regular $\lambda < \kappa$ let CUB^κ_λ denote the set of $f : \kappa \to \kappa$ such that $\{\alpha < \kappa \mid f(\alpha) = 0\}$ contains a λ-closed unbounded subset. Then CUB^κ_λ does not have the BP.

Even Δ^1_1 sets can fail to have the BP:

Theorem

(a) In L, CUB^κ_λ is not Δ^1_1 for any λ but there are Δ^1_1 sets without the BP.
(b) CUB^κ_λ is consistently Δ^1_1 (Mekler-Shelah for $\kappa = \omega_1$, Hyttinen-Rautila whenever $\lambda^+ = \kappa$, SDF when $\lambda^+ < \kappa$).

Higher Descriptive Set Theory: Regularity Properties

A bit of good news:

Theorem

(Sam Coskey and SDF, independently) You can force Δ^1_1 sets to have the BP.

Perfect Set Property

A subset of κ^κ is *perfect* iff it is the set of branches through a subtree of $\kappa^{<\kappa}$ which has no isolated branches and is $< \kappa$-closed

X has the *perfect set property (PSP)* iff it either has size at most κ or contains a perfect subset

Open sets trivially have the PSP

Higher Descriptive Set Theory: Regularity Properties

As Mekler-Väänänen observed, you need an inaccessible to get the PSP for closed sets, because you need to kill κ-Kurepa trees

Theorem

In L, the PSP fails for closed sets (for all κ).

This is because in L there is a "quasï"-Kurepa tree at every regular κ

Theorem

(Philipp Schlicht and SDF, independently) After converting an inaccessible into ω_2 with an ω-closed Lévy collapse, the PSP holds for all Σ^1_1 sets.

Question: Is the PSP for Π^1_1 sets consistent?

Higher Descriptive Set Theory: Borel Reducibility

We need to generalise the theory of Borel reducibility from ω to κ

A function $f : X_0 \to X_1$ where X_0, X_1 are Borel subsets of κ^κ is a *Borel function* iff $f^{-1}[Y]$ is Borel whenever Y is Borel

Let E_0, E_1 be equivalence relations on Borel subsets X_0, X_1 of κ^κ.

$E_0 \leq_B E_1$ (E_0 is *Borel reducible* to E_1) iff for some Borel function $f : X_0 \to X_1$:

$$x_0 E_0 y_0 \text{ iff } f(x_0) E_1 f(y_0)$$

Higher Descriptive Set Theory: Borel Reducibility

Now recall the following picture from the classical case:

$1 <_B 2 <_B \cdots <_B \omega <_B \text{id} <_B E_0$

forms an initial segment of the Borel equivalence relations under $\leq_B$ where n denotes an equivalence relation with n classes for $n \leq \omega$, id denotes equality on ω^ω and E_0 denotes equality modulo finite on ω^ω

At κ we easily get the initial segment

$1 <_B 2 <_B \cdots <_B \omega <_B \omega_1 <_B \cdots <_B \kappa$

(Silver Dichotomy) Can id (equality on κ^κ) be the successor of κ?

This implies that Borel sets have the PSP, so it fails in L and its consistency requires an inaccessible

Higher Descriptive Set Theory: Borel Reducibility

(Glimm-Effros) Can E_0 be the successor of id (at κ)?

Versions of E_0:
For regular $\lambda \leq \kappa$, define $E_0^{<\lambda}$ = equality modulo sets of size $< \lambda$

Fact: For $\lambda < \kappa$, $E_0^{<\lambda}$ is Borel bireducible with id

So we can forget about $E_0^{<\lambda}$ for $\lambda < \kappa$ and set $E_0 = E_0^{\kappa}$, equality modulo bounded

Higher Descriptive Set Theory: Borel Reducibility

Other versions of E_0:

For regular $\lambda < \kappa$ define E^κ_λ = equality modulo the ideal of λ-nonstationary sets

These equivalence relations are key for connecting Shelah Classification with Higher Descriptive Set Theory

Theorem

(SDF-Hyttinen-Kulikov) Relative to an inaccessible it is consistent that κ is inaccessible and the E^κ_λ are pairwise Borel-incomparable for distinct regular $\lambda < \kappa$. And relative to a weak compact it is consistent that $E^{\omega_2}_\omega$ is Borel-reducible to $E^{\omega_2}_{\omega_1}$.

Higher Descriptive Set Theory: Borel Reducibility

Are there Borel-incomparable Borel equivalence relations? We do have:

Theorem

(SDF-Hyttinen-Kulikov) It is consistent to have an embedding from $(\mathcal{P}(\kappa), \subseteq)$ *into the ordering of* Δ^1_1 *equivalence relations under Borel reducibility.*

Shelah Classification and Higher Descriptive Set Theory

We now connect Shelah Classification with Higher Descriptive Set Theory.

For simplicity assume GCH and $\kappa = \lambda^+$ where λ is uncountable and regular.
$\mathrm{Isom}^{\kappa}_{T}$ is the isomorphism relation on the models of T of size κ.

Theorem

(SDF-Hyttinen-Kulikov)
(a) T is classifiable and shallow iff $\mathrm{Isom}^{\kappa}_{T}$ is Borel.
(b) T is classifiable iff for all regular $\mu < \kappa$, $E_{S^{\kappa}_{\mu}}$ is not Borel reducible to $\mathrm{Isom}^{\kappa}_{T}$.
(c) In L, T is classifiable iff $\mathrm{Isom}^{\kappa}_{T}$ is Δ^1_1.

The proof uses Ehrenfeucht-Fraissé games:

The Game $EF^{\kappa}_{t}(\mathcal{A},\mathcal{B})$

$\mathcal{A}$, $\mathcal{B}$ are structures of size κ, t is a tree.
Player I chooses size $< \kappa$ subsets of $A \cup B$ and player II builds a partial isomorphism between $\mathcal{A}$ and $\mathcal{B}$ which includes these sets.
The moves take place along a branch through the tree t.
Player II wins iff he survives until a cofinal branch is reached.

The tree t *captures* $Isom^{\kappa}_{T}$ iff for all size κ models $\mathcal{A}$, $\mathcal{B}$ of T, $\mathcal{A} \simeq \mathcal{B}$ iff Player II has a winning strategy in $\mathrm{EF}^{\kappa}_{t}(\mathcal{A},\mathcal{B})$.

Now there are 4 cases:

Case 1: T is classifiable and shallow.

Then Shelah's work shows that some well-founded tree captures Isom^κ_T. We use this to show that Isom^κ_T is Borel.

Shelah Classification and Higher Descriptive Set Theory

Case 2: T it classifiable and deep.

Then Shelah's work shows that no fixed well-founded tree captures Isom^κ_T. We use this to show that Isom^κ_T is not Borel.

Shelah's work also shows that $L_{\infty\kappa}$ equivalent models of T of size κ are isomorphic. This means that the tree $t = \omega$ (with a single infinite branch) captures Isom^κ_T. As the games $\mathrm{EF}^\kappa_\omega(\mathcal{A}, \mathcal{B})$ are determined, this shows that Isom^κ_T is Δ^1_1.

We must also show: $E_{S^\kappa_\mu}$ (equality modulo the μ-nonstationary ideal) is not Borel reducible to Isom^κ_T for any regular $\mu < \kappa$. This is because (in this case) Isom^κ_T is absolutely Δ^1_1, whereas μ-stationarity is not.

Shelah Classification and Higher Descriptive Set Theory

Now we look at the unclassifiable cases.

Recall: Classifiable means superstable without DOP and without OTOP.

Case 3: T is unstable, superstable with DOP or superstable with OTOP.

Work of Hyttinen-Shelah and Hyttinen-Tuuri shows that in this case no tree of size κ without branches of length κ captures Isom^κ_T. This can be used to show Isom^κ_T is not Δ^1_1.

But $E_{S^\kappa_\lambda} \leq_B \text{Isom}^\kappa_T$ is harder.
Following Shelah, there is a Borel map $S \mapsto \mathcal{A}(S)$ from subsets of κ to Ehrenfeucht-Mostowski models of T built on linear orders so that $\mathcal{A}(S_0) \simeq \mathcal{A}(S_1)$ iff $S_0 = S_1$ modulo the λ-nonstationary ideal.

Shelah Classification and Higher Descriptive Set Theory

Case 4: T is stable but not superstable.

This is the hardest case and requires some new model theory. Hyttinen replaces Ehrenfeucht-Mostowski models built on linear orders with primary models built on trees of height $\omega + 1$ to show $E_{S^\kappa_\omega} \leq_B \mathrm{Isom}^\kappa_T$. (We don't know if $E_{S^\kappa_\lambda} \leq_B \mathrm{Isom}^\kappa_T$ or if Isom^κ_T could be Δ^1_1 in this case.)

Now we have all we need to prove the Theorem mentioned earlier:

(a) T is classifiable and shallow iff Isom^κ_T is Borel.

We showed that if T is classifiable and shallow then Isom^κ_T is Borel and if it is classifiable and deep it is not. If T is not classifiable then some $E_{S^\kappa_\mu}$ Borel reduces to Isom^κ_T, so the latter cannot be Borel.

(b) T is classifiable iff for all regular $\mu<\kappa$, $E_{S^\kappa_\mu}$ is not Borel reducible to Isom^κ_T.

We showed that if T is not classifiable then $E_{S^\kappa_\mu}$ is Borel reducible to Isom^κ_T where μ is either λ or ω. We also showed that if T is classifiable and deep then no $E_{S^\kappa_\mu}$ is Borel reducible to Isom^κ_T, by an absoluteness argument. When T is classifiable and shallow there is no such reduction as Isom^κ_T is Borel.

(c) In L, T is classifiable iff Isom^κ_T is Δ^1_1.

We showed that if T is classifiable then Isom^κ_T is Δ^1_1, in ZFC.
If T is not classifiable then $E_{S^\kappa_\mu}$ Borel reduces to Isom^κ_T for some μ, and in L, $E_{S^\kappa_\mu}$ is not Δ^1_1.

Shelah Classification and Higher Descriptive Set Theory: Open Problems

Regularity Properties at uncountable regular cardinals

1. Is the PSP for Π^1_1 consistent?
2. Investigate other regularity properties.

Borel Reducibility at uncountable regular cardinals

3. Are there incomparable Borel equivalence relations?
4. Are the Silver or Glimm-Effros Dichotomies for Borel equivalence relations consistent? Do they hold for isomorphism relations?
5. Are there Σ^1_1 equivalence relations which are not Borel reducible to graph isomorphism?

Shelah Classification and Higher DST

6. Can Isom^κ_T be Δ^1_1 for an unclassifiable T?
7. Does equality modulo the λ-nonstationary ideal Borel reduce to Isom^κ_T for stable, unsuperstable T $(\kappa = \lambda^+)$?