Extensions of Partial Colorings on Countable Groups

Su Gao

Department of Mathematics
University of North Texas

Luminy Set Theory Workshop
October 4–8, 2010

This is joint work with Steve Jackson and Brandon Seward.

A coloring property for countable groups, preprint, 2007.

Group colorings and Bernoulli subflows, in preparation.

Part I: Density of 2-colorings

In this first part we review the definitions and the density result for 2-colorings on countably infinite groups.

Free Bernoulli Subflows

Let G be a countable group.

Bernoulli G-flow: the G-space $2^G = \{0,1\}^G$ with the shift action

$$(g \cdot x)(h) = x(g^{-1}h)$$

subflow: closed invariant subset of 2^G

free subflow: closed invariant subset of $F(G)$, the free part of 2^G

For a countable group G and $x \in 2^G$, let

$$G_x = \{g \in G : g \cdot x = x\}.$$

x is aperiodic or free: G_x is trivial

$F(G)$, the free part of 2^G: the set of all aperiodic x

x is periodic: G_x is nontrivial

x is aperiodic iff any $y = g \cdot x \in [x]$ is aperiodic iff

for any $s \in G$ *with* $s \neq 1_G$,

$$\forall g \in G \; \exists t \in G \; x(gst) \neq x(gt)$$

Constructing free subflows

$\Longleftrightarrow$

constructing $x \in 2^G$ so that $\overline{[x]} \subseteq F(G)$

i.e., $x \in 2^G$ such that every $y \in \overline{[x]}$ is aperiodic

2-Colorings

Let G be a countable group. A 2-coloring on G is a function $x : G \to \{0, 1\}$ such that

for any $s \in G$ with $s \neq 1_G$, there is a finite set $T \subseteq G$ such that

$$\forall g \in G\ \exists t \in T\ x(gst) \neq x(gt).$$

Lemma (GJS, Pestov)
x is a 2-coloring on G iff $\overline{[x]}$ is a free subflow.

Blocking

Let G be a countable group and $1_G \neq s \in G$. A element $x \in 2^G$ **blocks** s if

there is a finite set $T \subseteq G$ such that

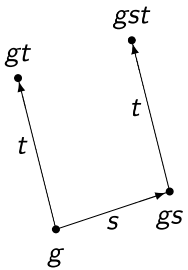
$$\forall g \in G\ \exists t \in T\ x(gst) \neq x(gt).$$

Lemma

x blocks s iff $s \notin G_z$ for any $z \in \overline{[x]}$.

there is a finite set $T \subseteq G$ *such that*

$$\forall g \in G\ \exists t \in T\ x(gst) \neq x(gt).$$



Theorem There is a 2-coloring on every countably infinite group.

Theorem For any countably infinite group G the set of all 2-colorings on G is dense.

Theorem (Restatement) For any countably infinite group G, any partial function on G with finite domain can be extended to a 2-coloring.

Extendability Problems

The Extendable Domain Problem

For which infinite domain A in a countably infinite group G is it true that any partial function on A can be extended to a 2-coloring?

Extendability Problems

The Automatic Extendability Problem

For which countably infinite group G is it true that any extension of a 2-coloring on a subgroup is automatically a 2-coloring on G?

Part II: The Fundamental Method for Constructing 2-colorings

In this part of the talk I will review the main ideas used in the proof of the construction of a 2-coloring for any countably infinite group. Variations of these ideas will be used to prove results about the extendability problems.

How to block a single element s

Idea 1: Develop a marker structure (Δ, F):

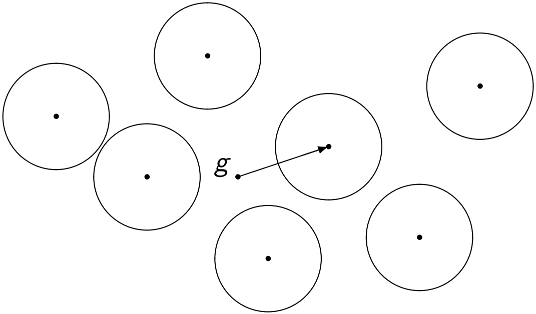
F: a finite set we refer to as the basic marker region

Δ: the set of marker points serving as centers of marker regions

Each marker region other than F itself is a translate of F, i.e., of the form γF where $\gamma \in \Delta$

The marker regions form a maximal disjoint family in G

How to block a single element s



Any $g \in G$ is "within FF^{-1}" of Δ, i.e., there is $t \in FF^{-1}$ such that

$$gt \in \Delta$$

How to block a single element s

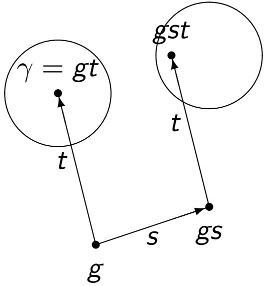
Idea 1: Develop a marker structure (Δ, F)
Idea 2: Develop a membership test for Δ points

A simple Δ-membership test would be a specific partial function $Q : A \to 2$, where $A \subseteq F$, such that

$$g \in \Delta \text{ iff } c(ga) = Q(a) \text{ for all } a \in A$$

How to block a single element s

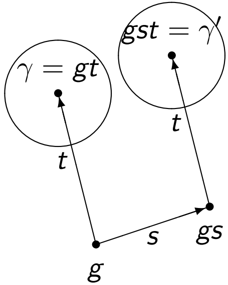
The argument for blocking would go as follows:



If $gst \notin \Delta$, then there is $a \in A$ such that

$$c(gta) \neq c(gsta)$$

If $gst \in \Delta$ then we have to keep working:



In this situation

$$\gamma' = gst = (gt)t^{-1}st = \gamma t^{-1}st$$

and so

$$\gamma^{-1}\gamma' \in FF^{-1}sFF^{-1}$$

How to block a single element s

Consider the graph $\Gamma = (\Delta, E)$ defined by

$$(\gamma, \gamma') \in E \text{ iff } \gamma^{-1}\gamma' \text{ or } \gamma'^{-1}\gamma \in FF^{-1}sFF^{-1}$$

Every vertex in Γ has degree at most

$$2|F|^4$$

There is a graph-theoretic coloring of vertices of Γ with $2|F|^4 + 1$ colors.

How to code these graph-theoretic colors into the $\{0,1\}$-coloring of elements of G?

How to block a single element s

Idea 1: Develop a marker structure (Δ, F)
Idea 2: Develop a membership test for Δ points (using partial colorings!)
Idea 3: Code $\log(2|F|^4+1)$ many digits into the coloring

In each marker region leave $\log(2|F|^4+1)$ many points uncolored when the membership test was developed. Now code the graph-theoretic colors using these points.

Need to make sure

$$|F| >> \log(2|F|^4+1)!$$

Blocking all non-identity s simultaneously

We need infinitely many layers of marker sets and regions with the following properties:

F_n: a finite "basic" marker region on the n-th layer

Δ_n: the n-th layer marker set serving as the centers of the marker regions

Each marker region other than F_n itself is a translate of F_n, i.e., of the form γF_n where $\gamma \in \Delta_n$

The marker regions $\{\gamma F_n : \gamma \in \Delta_n\}$ form a maximal disjoint family in G

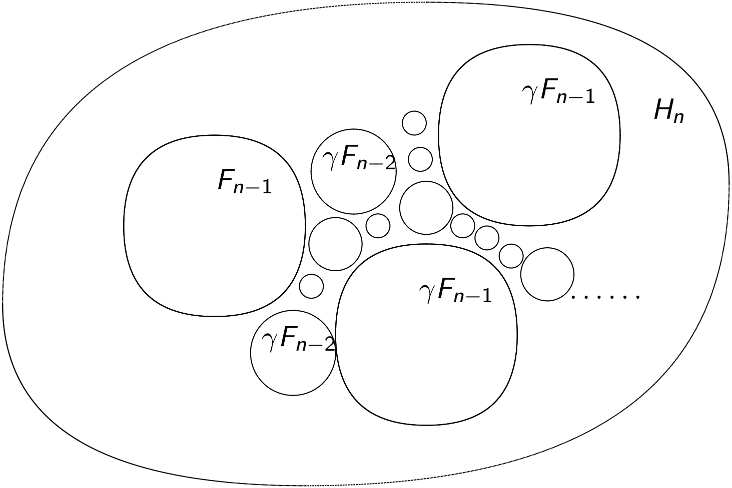
Cofinality: $\bigcup_n F_n F_n^{-1} = G$

Coherence: $F_n \subseteq F_{n+1}$, $\Delta_n \supseteq \Delta_{n+1}$; in fact, each F_{n+1} is the union of a number of disjoint translates of F_n with other disjoint successive translates of F_m, $m < n$

These are achieved by starting with preliminary finite but confinal regions

$$H_0 \subseteq H_1 \subseteq H_2 \subseteq \dots$$

with $\bigcup_n H_n = G$, and successively "filling" H_{n+1} by disjoint translates of F_m, $m \leq n$


γF_{n-1}
H_n
γF_{n-2}
F_{n-1}
γF_{n-1}
γF_{n-2}
......

Next we introduce a partial coloring c of G in such a way that elements of Δ_n can be detected by a membership test:

$$g \in \Delta_1 \iff \forall f \in \lambda_1 F_0 \; c(gf) = c(f)$$

for some fixed element $\lambda_1 \in F_1$

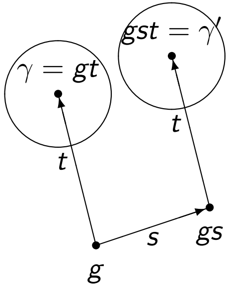
$$g \in \Delta_n \iff \forall f \in \Lambda_n \; c(gf) = c(f)$$

for some fixed finite set $\Lambda_n \subseteq F_n$

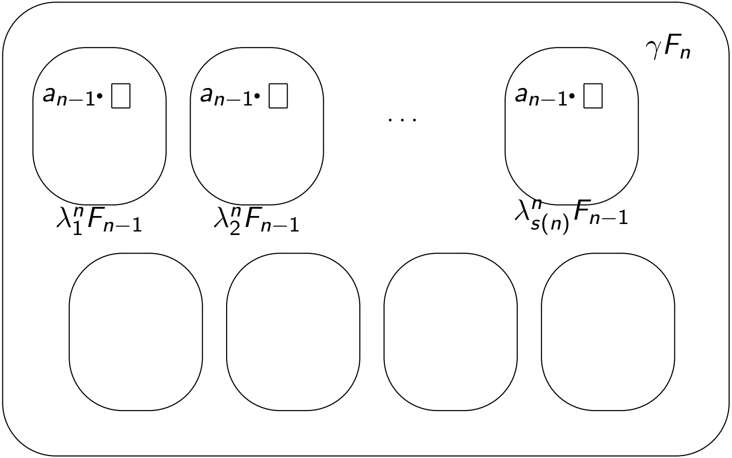
In particular, if $\gamma \in \Delta_n$ and $\eta \notin \Delta_n$, then there is some $f \in F_n$ such that

$$c(\gamma f) \neq c(\eta f)$$

We need to worry again about the situation:



The partial coloring c also has the property that there exist at least two elements $a_n, b_n \in F_n$ such that for any $\gamma \in \Delta_n$, $\gamma a_n, \gamma b_n \notin \mathrm{dom}(c)$, i.e, each marker region γF_n contains at least two uncolored elements to be colored at strategic positions


γF_n
$a_{n-1} \bullet$
$a_{n-1} \bullet$
$\cdots$
$a_{n-1} \bullet$
$\lambda_1^n F_{n-1}$
$\lambda_2^n F_{n-1}$
$\lambda_{s(n)}^n F_{n-1}$

To implement Idea 3 we make sure that

for any two elements $\gamma, \gamma' \in \Delta_n$ with $\gamma^{-1}\gamma \in F_n F_n^{-1} H_n F_n F_n^{-1}$, some of the "$a_{n-1}$" points in γF_n and in $\gamma' F_n$ are colored differently.

This is achieved by

- making sure there are enough copies of F_{n-1} in F_n (so that $2^{s(n)} > |F_n|^5$)
- pairs of marker points related as above are assigned different binary labels of length $s(n)$

Density

Density of 2-colorings is proved by noting

Any partial function with finite domain can be extended to a locally recognizable function.

Other results on extendability

Fact For any countably infinite group G there is a partial function c with $\mathrm{dom}(c)$ infinite and coinfinite so that any extension of c is a 2-coloring.

In contrast....

Fact For any countably infinite group G there is a partial function c with $\mathrm{dom}(c)$ infinite and coinfinite so that any extension of c is not a 2-coloring.

Just make sure $\mathrm{dom}(c)$ contains a translate of any finite subset of G.

Part III: Solutions to the Extendability Problems

In this part of the talk we give the solutions to the extendability problems and illustrate the method used in their proofs.

The Extendable Domain Problem

The Extendable Domain Problem

For which infinite domain A in a countably infinite group G is it true that any partial function on A can be extended to a 2-coloring?

It is necessary that A cannot contain a translate of every finite set , i.e.,

there is a finite $F \subseteq G$ such that for all $g \in G$, $gF \not\subseteq A$.

We call such sets slender.

Theorem

The following are equivalent for any countably infinite group G and $A \subseteq G$:

(i) Any partial function $c : A \to 2$ can be extended to a 2-coloring on G.

(ii) $1 \in 2^A$ can be extended to a 2-coloring on G.

(iii) Any partial function $c : A \to 2$ can be extended to perfectly many pairwise orthogonal 2-colorings on G.

(iv) A is slender.

(v) $1(\in 2^G) \notin \overline{[\chi_A]}$.

(vi) $1 \perp \chi_A$.

Theorem

Let G be a countably infinite group, $A \subseteq G$ slender, $s \in G - \{1_G\}$, and $y \in 2^A$. Then there exist $A' \supseteq A$ and $x \in 2^{A'}$ such that

(a) A' is slender;

(b) $x \supseteq y$;

(c) Any extension of x blocks s.

Furthermore, there are $x_0, x_1 \in 2^{A'}$ such that

(d) $x_0, x_1 \supseteq x$;

(e) $x_0 \perp x_1$.

The main idea of the proof is to develop a membership test for some marker set Δ with some background colors on A given!

This is significantly harder than the situation where the background is blank.

The membership test will be based on counting the number of 1's in a surrounding region, but

- it will not be a simple membership test (more than one pattern is possible)
- a lot more coding is needed

To illustrate how to use the slenderness condition, consider a finite set $B \subseteq G$ such that

for any $g \in G$, $|gB - A| \geq 2$.

The counting function is defined as

$$r_z(g) = |\{h \in B \,:\, gh \in \mathrm{dom}(z) \wedge z(gh) = 1\}|.$$

Thus $r_y(g) \leq |B| - 2$ for all $g \in G$.

Imagine the situation: for some $z \supseteq y$ and $g \in G$, $r_z(g) = |B|$. Then g can be distinguished from the background (to some extent), especially if we make sure there are no further points in gBB^{-1} being colored 1 beyond y.

Let $N = \max\{r_y(g) \,:\, g \in G\}$.

The membership test will be developed parallel to our definition of the marker structure (Δ, F).

Let $C = C^{-1}$ contain BB^{-1}. We will require $F >> C$, and the intention is to add at most 2 new points with color 1 within any translate gC of C, and they are all added within gB.

Still, to distinguish one point of C from another, we introduce a coding mechanism:
Define

$$\begin{array}{ccccc} C & \times & C & \to & G \\ (c_1 & , & c_2) & \mapsto & a(c_1, c_2) \end{array}$$

such that

$$a(c_1, c_2) = a(c_2, c_1)$$
$$c_1 a_1(c_1, c_2) C \cap C = \emptyset$$
$$c_1 a(c_1, c_2) C \cap c_3 a(c_3, c_4) C = \emptyset$$

if $\{c_1, c_2\} \neq \{c_3, c_4\}$.

Let

$$V = \{a(c_1, c_2) \, : \, c_1, c_2 \in C\}C$$

and

$$F \supseteq C^3 \cup CV$$

so that

$$\rho(F; C) \geq 2|C|^6 + \log(2|C|^{3N+3}|F|^2).$$

Let $D_0 = r_y^{-1}(N)$ be a maximal subset of $r_y^{-1}(N)$ with D_0-translates of CF disjoint.

Let $D_1 = r_y^{-1}(N-1)$ be a maximal subset of $r_y^{-1}(N-1)$ with D_1-translates of CF disjoint and

$$D_1 C^4 F \cap D_0 CF = \emptyset.$$

In general, let $D_m \subseteq r_y^{-1}(N-m)$ be a maximal subset of $r_y^{-1}(N-m)$ with D_m-translates of CF disjoint and

$$D_m C^{3m+1} F \cap \bigcup_{0 \leq i < m} D_i CF = \emptyset.$$

Let $D = \bigcup_{0 \leq m \leq N} D_m$.

We define an extension y' of y such that

$$r_y(g) \leq r_{y'}(g) \leq r_y(g) + 2$$

$$r_{y'}(g) > r_y(g) + 1 \Rightarrow g \in DC$$

$$r_{y'}(g) > r_y(g) \Rightarrow g \in D(C \cup V)$$

and for any $d \in D$ and $c_1 \neq c_2 \in C$,

$$((dc_1)^{-1} \cdot y') \restriction V \neq ((dc_2)^{-1} \cdot y') \restriction V.$$

This allows us to impose a total ordering $\prec$ of 2^V on dC for each $d \in D$.

For any $d \in D_m$, let $dc \in \Delta_m$ iff

c is the unique element of

$$S = \{c' \in C \,:\, r_{y'}(dc') = N - m + 2\}$$

so that $((dc)^{-1} \cdot y') \upharpoonright V$ is the $\prec$-largest among

$$\{((dc')^{-1} \cdot y') \upharpoonright V \,:\, c' \in S\}.$$

Let $\Delta = \bigcup_{0 \le m \le N} \Delta_m$.

Some more coding (details omitted here) is employed to go back and forth between elements of D and those of Δ.

The resulting membership test:

$g \in \Delta_0$ iff

$$r_z(g) = N + 2 \textit{ and } h \prec_z g \textit{ for all } h \in gC^2 \textit{ with } r_z(h) = N + 2$$

$g \in \Delta_m$ iff

there is $c \in C$ *such that*

$$r_z(g) = N - m + 2$$

$$r_z(gc) = N - m + 2$$

$$gcC^{3m+1}F \cap \bigcup_{1 \le i < m} D_i CF = \emptyset$$

$$h \prec_z g \text{ for all } h \in gC^2 \text{ with } r_z(h) = N - m + 2$$

The Automatic Extendability Problem

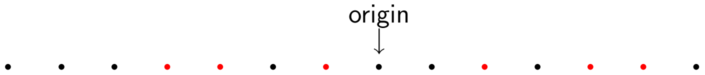
For which countably infinite group G is it true that any extension of a 2-coloring on a subgroup is automatically a 2-coloring on G?

Answer

$\mathbb{Z}$ is the only such group.

Example

Take a 2-coloring on $\mathbb{Z}$



0: red, 1: black

Space them out

$\circ$: undefined

Automatic extendability

Theorem

If countably infinite group G has the property that any extension of a 2-coloring on a subgroup is a 2-coloring on G, then $G \cong \mathbb{Z}$.

If G is such a group then

- every nontrivial subgroup has finite index
- $\mathrm{Z}(G) \cong \mathbb{Z}$
- $G/\mathrm{Z}(G)$ is finite
- Any Sylow subgroup of $G/\mathrm{Z}(G)$ is cyclic
- Using a theorem of Hölder, Burnside, and Zassenhaus characterizing all finite groups whose Sylow subgroups are cyclic, $G \cong \mathbb{Z}$

Reference

Results presented in this talk are available as a chapter in the upcoming monograph

Group Colorings and Bernoulli Subflows

Chapter 10. Extensions of Partial Colorings.