2-dimensional convexity revisited

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Convexity numbers, cliques, and the Kubiś set

Definition

Let $S \subseteq \mathbb{R}^n$. The *convexity number* $\gamma(S)$ of S is the least size of of a family $\mathcal{F}$ of convex sets such that $S = \bigcup \mathcal{F}$.

S is *countably convex* if $\gamma(S) \leq \aleph_0$ and otherwise *uncountably convex*.

A set $A \subseteq S$ is *defected* in S if the convex hull of A is not a subset of S.

A set $C \subseteq S$ is an *m-clique* of S if all m-element subsets of C are defected in S.

Remark

By Caratheodory's theorem, the convex structure of a set $S \subseteq \mathbb{R}^n$ is determined by the $(n+1)$-uniform *defectedness hypergraph*

$$G(S) = (S, \{A \in [S]^n : A \text{ is defected in } S\}).$$

The convexity number $\gamma(S)$ is the chromatic number of $G(S)$. An $(n+1)$-clique in S is a clique in $G(S)$.

The size of an infinite *m*-clique in $G(S)$ is a lower bound of $\gamma(S)$.

Theorem (Folklore?)

If a closed set $S \subseteq \mathbb{R}^n$ has an uncountable m-clique for any $m \in \omega$, then it has a perfect $(n+1)$-clique.

Remark

If $S \subseteq \mathbb{R}^n$ is closed, then (the edge relation of) $G(S)$ is open.

Since the Open Coloring Axiom holds for closed subsets of $\mathbb{R}$, every closed subset of $\mathbb{R}$ is either countably convex or has a perfect 2-clique.

Definition

For all $\{x, y\} \in [\omega^\omega]^2$ let

$$\Delta(x, y) = \min\{n \in \omega : x(n) \neq y(n)\}$$

and

$$c_{\min}(x, y) = \Delta(x, y) \mod 2.$$

Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ be a function. We say that $\{x, y\} \in [A]^2$ with $x < y$ is in *configuration* $\sqcap$ (respectively $\sqcup$) if for all $z \in (x, y)$, $(z, f(z))$ is either on or strictly above (below) the line segment joining $(x, f(x))$ and $(y, f(y))$.

Theorem (Kubiś)

There are a closed set $S \subseteq \mathbb{R}^2$, a topological embedding $e : 2^\omega \to \mathbb{R}$, and a differentiable function $f : e[2^\omega] \to \mathbb{R}$ with the following properties:

1. *For all $\{x, y\} \in [2^\omega]^2$, $\{e(x), e(y)\}$ is of configuration $\sqcap$ if $c_{\min}(x, y) = 1$ and of configuration $\sqcup$ otherwise.*
2. *$S \setminus f$ is countably convex.*
3. *For $x \in 2^\omega$ let $g(x) = (e(x), f(e(x)))$. Then for all $A \subseteq 2^\omega$, the set $g[A]$ is defected in S iff A is homogeneous with respect to $c_{\min}$.*

The set S is uncountably convex and does not have an uncountable 3-clique.

The Decomposition Theorem

Theorem (Kubiś)

Let $S \subseteq \mathbb{R}^2$ be closed, uncountably convex, and without a perfect 3-clique. Then there are a countably convex set $A \subseteq \mathbb{R}^2$ and a sequence $(B_n)_{n\in\omega}$ of G_δ-sets such that

$$S = A \cup \bigcup_{n\in\omega} B_n$$

and for each $n \in \omega$ there is a continuous coloring $c_n : [B_n]^2 \to 2$ such that $B \subseteq B_n$ is not defected in S iff B is homogeneous wrt c_n.

Here the sets B_n are affinely isomorphic to graphs of Lipschitz functions and the colorings are colorings by configuration.

Lemma (Transitivity)

Let $C \subseteq \mathbb{R}$, let $f : C \to \mathbb{R}$ be a function such that every two-element set $\{x, y\} \subseteq C$ has a configuration, and let $c : [C]^2 \to \{\sqcup, \sqcap\}$ be the coloring that assigns to each pair its configuration.

a) Let $x_1, x_2, x_3 \in C$ be such that $x_1 < x_2 < x_3$. If $c_K(x_1, x_2) = c_K(x_2, x_3) = \sqcap$, then $c_K(x_1, x_3) = \sqcap$. If $c_K(x_1, x_2) = c_K(x_2, x_3) = \sqcup$, then $c_K(x_1, x_3) = \sqcup$.

b) Let $x_1, x_2, x_3, x_4 \in C$ be such that $x_1 < x_2 < x_3 < x_4$. If $c_K(x_1, x_3) = c_K(x_2, x_4) = \sqcap$, then $c_K(x_1, x_4) = \sqcap$. If $c_K(x_1, x_3) = c_K(x_2, x_4) = \sqcup$, then $c_K(x_1, x_4) = \sqcup$.

Definition

A graph $G = (V, E)$ is P_4-*free* if it does not contain an induced copy of the path of length 3 on 4 vertices.

Theorem

Let $C \subseteq \mathbb{R}$, let $f : C \to \mathbb{R}$ be a function such that every two-element set $\{x, y\} \subseteq C$ has a configuration. Let G be the graph on the set C of vertices where $\{x, y\}$ is an edge iff $\{x, y\}$ is in configuration $\sqcap$. Then G is P_4-free.

In particular, G is perfect (in the graph-theoretic sense).

Homogeneity numbers

Definition

Let X be a Polish space and let $c : [X]^2 \to 2$ be a continuous coloring. The *homogeneity number* $\mathfrak{hm}(c)$ is the least size of a family of homogeneous subsets of X that covers all of X.

The coloring c is *uncountably homogeneous* if $\mathfrak{hm}(c) > \aleph_0$.

Lemma (G., Kojman)

A continuous coloring $c : [X]^2 \to 2$ on a Polish space X is uncountably homogeneous iff there is a topological embedding $e : 2^\omega \to X$ such that for all $\{x, y\} \in [X]^2$,

$$c_{\min}(x, y) = c(e(x), e(y)).$$

In particular, the homogeneity number $\mathfrak{hm} = \mathfrak{hm}(c_{\min})$ is minimal among all uncountable homogeneity numbers of continuous colorings on Polish spaces.

Theorem

a) $\mathfrak{hm}^+ \geq 2^{\aleph_0}$

b) $\mathfrak{hm}$ *is an upper bound for all cardinal invariants in Cichoń's diagram.*

c) There is a continuous coloring $c_{\max} : [2^\omega]^2 \to 2$ *whose homogeneity number is maximal among all homogeneity numbers of continuous colorings on Polish spaces.*

d) It is consistent that $\mathfrak{hm}(c_{\max}) < 2^{\aleph_0}$ *(G., Schipperus).*

e) It is consistent that $\mathfrak{hm} < \mathfrak{hm}(c_{\max})$ *(G., Goldstern, Kojman).*

Remark

In the model of $\mathfrak{hm} < \mathfrak{hm}(c_{\max})$, the perfect continuous colorings have homogeneity numbers equal to $\mathfrak{hm}$.

By the Decomposition Theorem every uncountably convex, closed set $S \subseteq \mathbb{R}^2$ without a perfect 3-clique has $\gamma(S) = \mathfrak{hm}(c)$ for some P_4-free continuous coloring on a Polish space.

Corollary

It is consistent that $\gamma(S) < \mathfrak{hm}(c_{\max})$ holds for every closed set $S \subseteq \mathbb{R}^2$ without a perfect 3-clique.

P_4-free continuous colorings

Theorem (Seinsche)

The class of finite P_4-free graphs is the smallest class of graphs that contains the graph on a single vertex and is closed under complementation and disjoint union.

Corollary

A finite graph G is P_4-free iff it embeds into $G_{\min} = (2^\omega, c_{\min}^{-1}(1))$.

Theorem

Let $c : [X]^2 \to 2$ be a continuous coloring on a Polish space. If c is P_4-free, then X is the union of not more than $\mathfrak{hm}$ sets $A \subseteq X$ such that $c \restriction [A]^2$ embeds into $c_{\min}$.

Corollary

If $c : [X]^2 \to 2$ is an uncountably homogeneous, P_4-free, continuous coloring on a Polish space, then $\mathfrak{hm}(c) = \mathfrak{hm}$.

Corollary

If $S \subseteq \mathbb{R}^2$ is closed, uncountably convex, and does not contain a perfect 3-clique, then $\gamma(S) = \mathfrak{hm}$.

Thank you!