Countable linear orders in reverse mathematics

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Reverse mathematics *is a program in mathematical logic that seeks to determine which axioms are required to prove theorems of mathematics. The method can briefly be described as "going backwards from the theorems to the axioms". This contrasts with the ordinary mathematical practice of deriving theorems from axioms.*

Wikipedia

Work over a weak base system. Various standard axioms provide strengthening. Given a theorem Φ, find, ideally, a standard axiom A so that, over the base system:

1. A is enough to prove Φ.
2. A is necessary to prove Φ.

Primary reference, Simpson [1999].

Reverse mathematics, continued

Some theorems addressed by reverse mathematics:

- Heine-Borel theorem on $[0, 1]$.
- Sequential completeness of $\mathbb{R}$.
- Bolzano–Weierstrass theorem.
- The perfect set theorem.
- Open determinacy.
- Cantor-Bendixson theorem.
- $\ldots$.

Natural base system RCA_0: axioms of PA for the natural numbers with only Σ^0_1 induction, and Δ^0_1 comprehension.

Additional axioms provide *sets* of natural numbers, beyond the recursive sets one gets from the base system.

Reverse mathematics measures how much of this extra strength is needed for each theorem.

Subsystems of analysis

Standard markers of strength include the following set existence axioms, increasing over the base system RCA_0, consisting of PA^-, Σ^0_1 induction, and Δ^0_1 comprehension.

1. Δ^0_1 comprehension: for Σ^0_1 formulas φ, ψ, if $\varphi(n) \leftrightarrow \neg\psi(n)$, then $\{n \mid \varphi(n)\}$ exists.
2. Weak König lemma: each infinite subtree of the binary tree has a branch.
3. Arithmetic comprehension.
4. Weak Σ^1_1 choice: if $(\forall n)(\exists! x)\varphi(n, x)$, then there is $\langle y_n \mid n < \omega\rangle$ so that $(\forall n)\varphi(n, y_n)$. Arithmetic φ
5. Δ^1_1 comprehension.
6. Σ^1_1 choice: as 4, but without assuming uniqueness.
7. Arithmetic transfinite recursion: arithmetic comprehension can be iterated transfinitely.
8. Π^1_1 comprehension.
9. Π^1_2 comprehension.

Subsystems of analysis, continued

1. Δ^0_1 comprehension.
2. Weak König lemma.
3. Arithmetic comprehension.
4. Weak Σ^1_1 choice. (With uniqueness.)
5. Δ^1_1 comprehension.
6. Σ^1_1 choice.
7. Arithmetic transfinite recursion.
8. Π^1_1 comprehension.
9. Π^1_2 comprehension.

Added to RCA_0, forming subsystems of analysis.

1, 2, 3, 7, 8 give big five systems of reverse mathematics.

4, 5, 6 give systems of hyperarithmetic analysis:

T is a theory of hyperarithmetic analysis if (a) its ω models are closed under joins and hyperarithmetic reducibility; (b) it holds in $\mathrm{HYP}(x)$ for all x.

Connection with Set Theory

Several results in reverse mathematics at the level of hyperarithmetic theories use set theory, particularly Steel [1977] forcing.

Theorem (Steel [1977])

Δ^1_1 comprehension does not imply Σ^1_1 choice.

Theorem (Van Wesep [1977])

Weak Σ^1_1 choice does not imply Δ^1_1 comprehension.

More recently additional axioms introduced by Montalbán.

Theorem (Montalbán [2008], [2006])

Π^1_1 separation is strictly between Δ^1_1 comprehension and Σ^1_1 choice. Other principles on "game comprehension" strictly below weak Σ^1_1 choice.

Steel forcing

Let $M = L_{\omega_1^{ck}}$. Force over M.

Fix a linear order $\prec$ in M whose wellfounded part is isomorphic to ω_1^{ck}.

Conditions are $\langle T, f, h\rangle$ where:

1. $T \subseteq \omega^{<\omega}$ is a finite tree. For $x, y \in T$ Write $x < y$ to mean x is an initial segment of y.
2. f is a finite partial function from ω into T. Let T_{wf} consist of nodes of T which are not in the downward closure of $\mathrm{image}(f)$.
3. h is a *tagging function* from T_{wf} into $\prec$, meaning that if $y > x$ then $h(y) \prec h(x)$.

Ordered in the natural way: $\langle T^*, f^*, h^*\rangle < \langle T, f, h\rangle$ if: $T^* \supseteq T$, $\mathrm{dom}(f^*) \supseteq \mathrm{dom}(f)$ and $f^*(n) \geq f(n)$ for each $n \in \mathrm{dom}(f)$, h^* extends h.

Steel forcing, continued

Generic filter G gives rise to:

1. A tree $T = T_G \subseteq \omega^{<\omega}$.
2. A set of infinite branches $b_i = f_G(i)$ $(i < \omega)$ through T.
3. A rank function h_G from T_{wf} into $\prec$.

Work with models $M[T, B]$ where B is a finite subset of $\{b_i \mid i < \omega\}$. Key lemmas:

Lemma

The only infinite branches through T in $M[T, B]$ are the ones in B.

Lemma

For B_1, B_2 disjoint, $M[T, B_1 \cup B_2]$ can in some sense be viewed as a forcing extension of $M[T, B_1]$.

Countable linear orders

Definition

- A linear order $(U; <_U)$ is scattered if it does not embed $\mathbb{Q}$.
- A gap in U is a partition of U into sets L and R, closed leftward and rightward respectively.
- A gap $\langle L, R \rangle$ is a decomposition of U if U does not embed into L, and does not embed into R.
- U is **in**decomposable if, for every gap $\langle L, R \rangle$, U embeds into either L or R.

INDEC

Recall, U is indecomposable if, for every gap $\langle L, R \rangle$, U embeds into either L or R.

Definition

U is indecomposable to the left if whenever $\langle L, R \rangle$ is a gap with $L \neq \emptyset$, U embeds into L. Indecomposability to the right defined similarly.

Theorem (Jullien [1969])

Suppose U is scattered and indecomposable. Then U is indecomposable to the left, or indecomposable to the right.

Theorem termed *INDEC* by Montalbán.

Hyperarithmetic analysis

Montalbán [2006] was led to INDEC working on strength of Fraïssé's conjecture $=$ Laver [1971] theorem.

Investigating INDEC, Montalbán:

1. Observed that the proof of INDEC uses only RCA_0 + Δ^1_1 comprehension. It follows that INDEC holds in $\mathrm{HYP}(X)$ for all X.
2. Proved that every ω model of INDEC is closed under the αth Turing jump, for each ordinal α in the model. It follows that ω models of INDEC are closed under hyperarithmetic reducibility.

INDEC is thus a theorem of hyperarithmetic analysis.
It is the first natural example of such a theorem.

Exact strength

1. Δ^0_1 comprehension.
2. Weak König lemma.
3. Arithmetic comprehension.
4. Weak Σ^1_1 choice. (With uniqueness.)
5. Δ^1_1 comprehension.
6. Σ^1_1 choice.
7. Arithmetic transfinite recursion.
8. Π^1_1 comprehension.
9. Π^1_2 comprehension.

The precise strength of INDEC remained open.

Does it imply Δ^1_1 comprehension? can it be proved from any weaker axiom?

Ideally in reverse mathematics, its strength should fit exactly with one of the logical axioms.

But it does not.

Exact strength, continued

1. Δ^0_1 comprehension.
2. Weak König lemma.
3. Arithmetic comprehension.
4. Weak Σ^1_1 choice. (With uniqueness.)
5. Δ^1_1 comprehension.
6. Σ^1_1 choice.
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9. Π^1_2 comprehension.

Theorem (Neeman [2008])

(In $\mathrm{RCA}_0 + \Sigma^1_1$ *induction.) INDEC implies weak* Σ^1_1 *choice.*

Theorem (Neeman [2008])

Weak Σ^1_1 *choice does not imply INDEC.*

Theorem (Neeman [2008])

INDEC does not imply Δ^1_1 *comprehension.*

Non-implications are set theoretic consistency results, and use Steel forcing.

Implication

Theorem

(In $\mathrm{RCA}_0 + \Sigma^1_1$ *induction.) INDEC implies weak* Σ^1_1 *choice.*

Proof sketch.

Suppose $(\forall n)(\exists! x)\varphi(n,x)$, arithmetic φ.

Construct, in $\mathrm{RCA}_0 + \Sigma^1_1$ induction, a linear order $(U; <_U)$ so that:

1. U is scattered.
2. $L^* = \{a \mid U$ embeds into $R_a\}$ and $R^* = \{a \mid U$ embeds into $L_a\}$ form a non-trivial gap in U.
3. This gap codes a sequence $\langle y_n \mid n < \omega\rangle$ so that $(\forall n)\varphi(n, y_n)$.

By INDEC, $\langle L^*, R^*\rangle$ exists, hence $\langle y_n \mid n < \omega\rangle$ exists. □

In proof of 2, need to know that for all $m < \omega$, $\langle y_n \mid n < m\rangle$ exists. This is where Σ^1_1 induction is used.

Induction

Proved the following reversal, in the base system RCA_0 + Σ^1_1 induction: INDEC implies weak Σ^1_1 choice.

Standard base system, RCA_0, has only Σ^0_1 induction.

There are uses of stronger induction in reverse mathematics. For example Δ^1_3 comprehension plus Σ^1_1 induction proves Δ^0_3 determinacy, and induction cannot be dropped.

Relatively rare, but there are a few other examples of similar nature. Induction is part of the strength.

Result on strength of INDEC is a reversal: shows Theorem X implies Axiom Y where neither X nor Y implies strong induction.

Reversals that use strong induction can generally be refined to use only Σ^0_1 induction.

This is not the case with INDEC.

Induction, continued

Theorem (Neeman [201?])

In $\mathrm{RCA}_0 + \Delta^1_1$ *induction, INDEC does not imply weak* Σ^1_1 *choice.*

First example of a provably necessary use of more than Σ^0_1 induction in a reversal.

Proof involves a combination of Steel forcing with a construction of a non-standard model.

Let $T = T_G$, $f = g_G$ and $h = h_G$ be given by a generic for Steel forcing.

For each n let i_n be least so that $f(i_n)(0) = \langle n, * \rangle$. Let $U = \{b_{i_n} \mid n < \omega\}$.

Let $\mathfrak{A}$ be a transitive model of set theory containing M, T, f, g. Fix $\mathfrak{A}^*$ a non-standard elementary extension.

Model for theorem is $\bigcup_{B \subseteq U^*,\ \text{of standard finite size}} M^*[T^*, B]$.

Questions

There are potentially other results on necessary use of induction.

A linear order U is extendible if every partial order P which does not embed U has a linearization which does not embed U.

For example ω^* is extendible, and its extendibility is just the statement that every wellfounded order has a wellfounded linearization.

2 is not extendible. If R is indecomposable to the right, and L to the left, then $R + L$ is not extendible.

JUL is the statement that U fails to be extendible iff it has an essential segment of the form 2 or $R + L$ (for R and L as above).

Questions, continued

Theorem (Montalbán [2006])

(In $\mathsf{RCA}_0 + \Sigma^1_1$ *induction.)* JUL *is equivalent to Fraïssé's conjecture.*

Right-to-left direction uses Σ^1_1 induction.

Not known if use is necessary.

Theorem (Montalbán–Miller)

(In $\mathsf{RCA}_0 + \Sigma^1_1$ *choice* $+ \Sigma^1_1$ *induction.) Extendibility of the rationals is equivalent to* ATR*.*

Both directions use Σ^1_1 induction.

Not known if use is necessary.

Bibliography

- Pierre Jullien.
  Contribution à l'étude des types d'ordre dispersés.
  PhD thesis, Marseille, 1969.
- Richard Laver
  On Fraïssé's order type conjecture.
  Ann. of Math. (2), 93:89–111, 1971.
- Antonio Montalbán.
  Equivalence between Fraïssé's conjecture and Jullien's theorem.
  Ann. Pure App. Log., 139(1):1–42, 2006.
- Antonio Montalbán.
  Indecomposable linear orderings and hyperarithmetic analysis.
  J. Math. Log., 6(1):89–120, 2006.
- Antonio Montalbán.
  On the Π^1_1 separation principle.
  Mathematical Logic Quarterly, 54(6):563–578, 2008.
- Itay Neeman.
  The strength of Jullien's indecomposability theorem.
  J. Math. Log. 8(2):93–119, 2008.
- Itay Neeman.
  A necessary use of Σ^1_1 induction in a reversal.
  J. Sym. Log. to appear.
- Stephen G. Simpson.
  Subsystems of Second Order Arithmetic.
  Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1999.
- John R. Steel.
  Determinateness and subsystems of analysis.
  PhD thesis, University of California, Berkeley, 1977.
- Robert Van Wesep.
  Subsystems of second-order arithmetic, and desc. set theory under the axiom of determinateness.
  PhD thesis, University of California, Berkeley, 1977.