A unified approach to higher Souslin trees constructions

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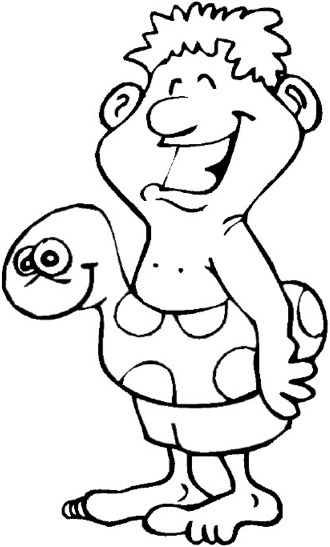
Introduction



Preliminaries: Combinatorial principles

Definition (Jensen, 1960's)

$\diamondsuit(S)$ asserts the existence of a sequence $\langle A_\alpha \mid \alpha \in S \rangle$ such that $\{\alpha \in S \mid A \cap \alpha = A_\alpha\}$ is stationary for every $A \subseteq \bigcup S$.

Definition (Jensen, 1960's)

$\square_\lambda$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \lambda^+ \rangle$ such that for all limit $\alpha < \lambda^+$:

- C_α is a club in α of order-type $\leq \lambda$;
- if $\beta \in \mathrm{acc}(C_\alpha)$, then $C_\alpha \cap \beta = C_\beta$.

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Definition (Schimmerling, 1995)

$\square_{\lambda,<\mu}$ asserts the existence of a sequence $\langle \mathcal{C}_\alpha \mid \alpha < \lambda^+ \rangle$ such that for all limit $\alpha < \lambda^+$:

- $0 < |\mathcal{C}_\alpha| < \mu$;
- C is a club in α of order-type $\leq \lambda$, for all $C \in \mathcal{C}_\alpha$;
- if $C \in \mathcal{C}_\alpha$ and $\beta \in \mathrm{acc}(C)$, then $C \cap \beta \in \mathcal{C}_\beta$.

It is convenient to write $\square_{\lambda,\mu}$ for $\square_{\lambda,<\mu^+}$. So, $\square_\lambda \equiv \square_{\lambda,1}$.

Preliminaries: λ^+-trees

Definition

- A λ^+-tree is a tree of height λ^+ whose levels are of size $\leq \lambda$;
- A λ^+-Aronszajn tree is a λ^+-tree having no branches of size λ^+;
- A λ^+-Souslin tree is a λ^+-Aronszajn tree having no antichains of size λ^+;

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- A λ^+-Souslin tree is a λ^+-Aronszajn tree having no antichains of size λ^+;
- A λ^+-tree is special if it is the union of λ many antichains.

Thus, a special tree is Aronszajn, and a Souslin tree is non-special.

The role of λ

Fact

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- There exists an ω_1-Aronszajn tree;
- if GCH holds, then for every regular cardinal λ, there exists a special λ^+-Aronszajn tree;
- GCH is consistent with the nonexistence of any λ^+-Aronszajn tree at some singular cardinal λ (modulo large cardinals);

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- if GCH holds, then for every regular cardinal λ, there exists a special λ^+-Aronszajn tree;
- GCH is consistent with the nonexistence of any λ^+-Aronszajn tree at some singular cardinal λ (modulo large cardinals);
- The existence of an ω_1-Souslin tree is independent of GCH;
- Any ω_1-Aronszajn tree can be made special in some cofinalities-preserving extension;
- If $V = L$, then for every uncountable cardinal λ, there exists a λ^+-Souslin tree which remains non-special in any cofinalities-preserving extension.

The role of λ (cont.)

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Many λ^+-Souslin trees constructions makes an explicit distinction between the case that λ is a regular cardinal and the case that λ is singular.

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Let us give two examples..

Example 1: Jensen's classical theorems

Theorem (Jensen, late 1960's)

Suppose that λ is a regular cardinal.
If $\lambda^{<\lambda} = \lambda$ and $\diamondsuit(E^{\lambda^+}_{\lambda})$ holds, then there exists a λ^+-Souslin tree.

Theorem (Jensen, early 1970's)

Suppose that λ is a singular cardinal.
If GCH is valid and $\square_\lambda$ holds, then there exists a λ^+-Souslin tree.

Example 2: Souslin trees which are hard to specialize

Theorem (Baumgartner, 1980's, building on Laver)

$GCH + \square_{\aleph_1}$ implies the existence of an $\aleph_2$-Souslin tree which remains non-special in any cofinalities-preserving extension.

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Theorem (Cummings, 1997)

Suppose that λ *is a singular cardinal of countable cofinality.*
If $\mathrm{CH}_\lambda + \square_\lambda$ *holds, and* $\mu^{\aleph_1} < \lambda$ *for all* $\mu < \lambda$*, then there exists a* λ^+*-Souslin tree which remains non-special in any c.p.e.*

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Theorem (Cummings, 1997)

Suppose that λ *is a singular cardinal of uncountable cofinality.*
If $\mathrm{CH}_\lambda + \square_\lambda$ *holds, and* $\mu^{\aleph_0} < \lambda$ *for all* $\mu < \lambda$*, then there exists a* λ^+*-Souslin tree which remains non-special in any c.p.e.*

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We shall propose a solution..

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2. Prove that this operational principle is a consequence of the "usual" hypotheses. Typically, this proof is divided into two or three independent subcases. However, this is done only once.

On clause 1

Ideally, the proposed principle would squeeze the most out of the prospective hypotheses (i.e., be logically equivalent to them).

The proposed proxy

For cardinals λ, μ, and a nonempty set of regular cardinals $\Gamma \subseteq \lambda^+$, we introduce the principle $\boxed{\diamondsuit}^{\Gamma}_{\lambda,<\mu}$, which combines $\square_{\lambda,<\mu}$ together with a reminiscent of $\diamondsuit(\lambda^+ \cap \mathrm{cof}(\Gamma))$. We then infer a λ^+-Souslin tree already from the weakest among these principles:

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Proposition

Suppose that λ is an uncountable cardinal.
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Proposition

Suppose that λ is an uncountable cardinal.
If $\boxtimes^{\Gamma}_{\lambda,\lambda}$ holds, then there exists a λ^+-Souslin tree.

Remarks

- The construction of the above tree is indeed uniform. That is, it does not depend on the identity of λ;
- Let κ denote the least cardinal such that $\lambda^\kappa > \lambda$. If $\Gamma \setminus \kappa \neq \emptyset$, then the resulting tree is moreover $(< \kappa)$-complete.

The principle $\boxdiamond^{\Gamma}_{\lambda,<\mu}$

$\boxdiamond^{\Gamma}_{\lambda,<\mu}$ is a rather weak statement and hence, somewhat lengthy.. We postpone the formal introduction of $\boxdiamond^{\Gamma}_{\lambda,<\mu}$. Instead, we mention the following:

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Fact (GCH)

In many cases, $\boxed{\diamondsuit}^{\{\theta\}}_{\lambda,<\mu}$ happens to be equivalent to the existence of a $\square_{\lambda,<\mu}$-sequence, $\langle \mathcal{C}_\alpha \mid \alpha < \lambda^+ \rangle$, with the additional property:

- for every unbounded $A \subseteq \lambda^+$, there exists some $\alpha \in E^{\lambda^+}_\theta$ such that $\mathrm{nacc}(C) \subseteq A$ for all $C \in \mathcal{C}_\alpha$.

Is the proposed principle $\boxdiamond^{\Gamma}_{\lambda,<\mu}$ any useful?

"Okay your father
managed to get a mouse.
Now how do we use it?"

Getting $\boxtimes^{\Gamma}_{\lambda,\lambda}$

Let λ denote a regular uncountable cardinal.

Theorem 1

If $\lambda^{<\lambda} = \lambda$ and $\diamondsuit(E^{\lambda^+}_{\lambda})$ holds, then $\boxtimes^{\{\lambda\}}_{\lambda,\lambda}$ holds.

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Corollary (Jensen, 1960's)

If $\lambda^{<\lambda} = \lambda$ and $\diamondsuit(E^{\lambda^+}_{\lambda})$ holds, then there exists a $(<\lambda)$-complete λ^+-Souslin tree.

Getting $\boxed{\diamondsuit}^{\Gamma}_{\lambda,\lambda}$ (cont.)

Let λ denote a regular uncountable cardinal.

Theorem 2

If $\lambda = 2^{<\lambda} < 2^{\lambda} = \lambda^{+}$ and there exists a non-reflecting stationary subset of $E^{\lambda^{+}}_{<\lambda}$, then $\boxed{\diamondsuit}^{\Gamma}_{\lambda,\lambda}$ holds.

Getting $\boxed{\diamondsuit}^{\Gamma}_{\lambda,\lambda}$ (cont.)

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Theorem 2

If $\lambda = 2^{<\lambda} < 2^{\lambda} = \lambda^{+}$ and there exists a non-reflecting stationary subset of $E^{\lambda^+}_{<\lambda}$, then $\boxed{\diamondsuit}^{\Gamma}_{\lambda,\lambda}$ holds.

Corollary (Gregory, 1976)

If $\lambda = 2^{<\lambda} < 2^{\lambda} = \lambda^{+}$ and there exists a non-reflecting stationary subset of $E^{\lambda^+}_{<\lambda}$, then there exists a λ^{+}-Souslin tree.

Getting $\boxtimes^{\Gamma}_{\lambda,\lambda}$ (cont.)

Theorem 3

If $2^{\aleph_0} = \aleph_1$ and NS_{ω_1} is saturated, then $\boxtimes^{\Gamma}_{\aleph_1,\aleph_1}$ holds.

Getting $\boxed{\diamondsuit}^{\Gamma}_{\lambda,\lambda}$ (cont.)

Theorem 3

If $2^{\aleph_0} = \aleph_1$ and NS_{ω_1} is saturated, then $\boxed{\diamondsuit}^{\Gamma}_{\aleph_1,\aleph_1}$ holds.

Corollary (Shelah, 1984)

If $2^{\aleph_0} = \aleph_1$ and NS_{ω_1} is saturated, then there exists an $\aleph_2$-Souslin tree.

Getting $\Diamond^{\Gamma}_{\lambda,\lambda}$ (cont.)

Let λ denote a singular cardinal.

Theorem 4

If GCH $+\square_{\lambda,<\mathrm{cf}(\lambda)}$ holds, then $\Diamond^{\Gamma}_{\lambda,\lambda}$ is valid.

Getting $\boxtimes^{\Gamma}_{\lambda,\lambda}$ (cont.)

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Corollary (Jensen, 1970's)

If GCH $+\square_{\lambda}$ holds, then there exists a λ^{+}-Souslin tree.

Corollary (Schimmerling, 2004)

If GCH $+\square_{\lambda,<\omega}$ holds, then there exists a λ^{+}-Souslin tree.

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Let us exemplify..

1. Souslin trees with a trivial automorphism group

Let λ denote an arbitrary uncountable cardinal.

Proposition

If $\boxed{\diamondsuit}^{\Gamma}_{\lambda,\lambda}$ holds, then there exists a rigid λ^+-Souslin tree.

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Immediate corollary

If any of the following is valid:

1. $\lambda^{<\lambda} = \lambda$ and $\diamondsuit(E^{\lambda^+}_{\lambda})$ holds
2. $\lambda^{<\lambda} = \lambda < \lambda^{\lambda} = \lambda^+$, and there exists a non-reflecting stationary subset of $E^{\lambda^+}_{<\lambda}$
3. $\lambda^{<\lambda} = \lambda = \kappa^+$ and $\mathrm{NS}_{\kappa^+} \restriction E^{\kappa^+}_{\kappa}$ is saturated
4. $\mathrm{GCH} + \square_{\lambda,<\mathrm{cf}(\lambda)}$ holds

then $\exists\ 2^{\lambda^+}$ many isomorphism types of rigid λ^+-Souslin trees.

2. Souslin trees which are hard to specialize

Let λ denote an arbitrary uncountable cardinal.

Proposition

If $\boxed{\Diamond}^{\Gamma}_{\lambda,1}$ holds, then there exists a λ^+-Souslin tree which remains non-special in any cofinalities-preserving extension.

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Let λ denote an arbitrary uncountable cardinal.

Proposition

If $\boxtimes^{\Gamma}_{\lambda,1}$ holds, then there exists a λ^+-Souslin tree which remains non-special in any cofinalities-preserving extension.
As always, the construction is uniform and we get as much completeness as possible.

Moreover

Suppose that $\kappa < \lambda = \lambda^{<\mu}$ are given infinite cardinals.
If $\boxtimes^{\Gamma}_{\lambda,1}$ holds with $\Gamma \setminus (\kappa \cup \mu) \neq \emptyset$, then $\exists$ a $(<\mu)$-complete λ^+-Souslin tree with a θ-ascent path for all regular $\theta \leq \kappa$.

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Theorem 5

The following are equivalent:

1. $\square_\lambda + \mathsf{CH}_\lambda$
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Corollary

The four Baumgartner and Cummings theorems.

3. Your contribution!

Pick your favorite $\Box$-based/$\Diamond$-based construction, and see if you can base it on $\boxed{\Diamond}^{\Gamma}_{\lambda,1}, \boxed{\Diamond}^{\Gamma}_{\lambda,<\omega}, \boxed{\Diamond}^{\Gamma}_{\lambda,\mathrm{cf}(\lambda)}$ or $\boxed{\Diamond}^{\Gamma}_{\lambda,\lambda}$.
An affirmative answer would make your construction portable in-between (successors of) regular and singular cardinals.

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For example, the proof of the following theorem is implicitly based on $\boxed{\Diamond}^{\Gamma}_{\lambda,1}$:

Theorem (Farah-Veličković, 2006)

Assume that $\square_\lambda + \mathrm{CH}_\lambda$ holds for a singular strong limit cardinal of uncountable cofinality λ.
Then there exists a complete Boolean algebra of size λ^+ which is ccc and weakly distributive, but is not a Maharam algebra.

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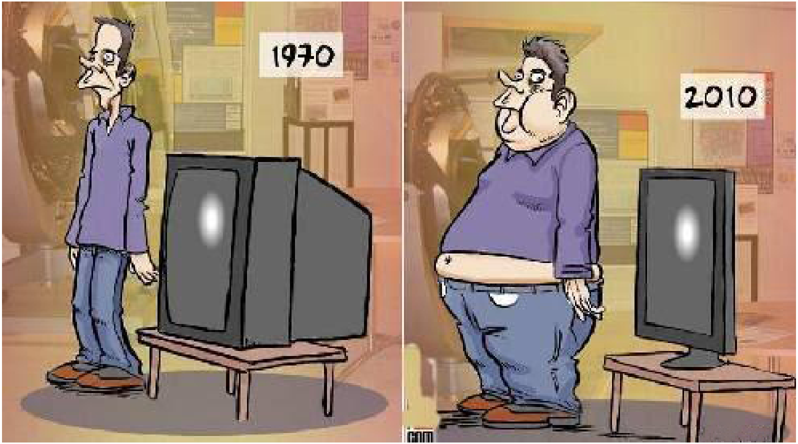
Theorem (Farah-Veličković), ported through Theorem 5

Assume that $\square_\lambda + \mathrm{CH}_\lambda$ holds for ~~a singular strong limit cardinal of uncountable cofinality λ~~ a cardinal $\lambda \geq \mathfrak{d}$.
Then there exists a complete Boolean algebra of size λ^+ which is ccc and weakly distributive, but is not a Maharam algebra.

Covering more recent trees constructions



Guessing of generalized clubs: $\lambda^*(\kappa, S)$

Let λ denote an uncountable cardinal.

Shelah's Club Guessing Theorem

If $S \subseteq E^{\lambda^+}_{<\lambda}$, then there exists a sequence $\langle C_\alpha \mid \alpha \in S\rangle$ such that:

1. C_α is a club in α for all $\alpha \in S$;
2. $\{\alpha \in S \mid C_\alpha \subseteq D\} \neq \emptyset$ for every club $D \subseteq \lambda^+$.

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König, Larson and Yoshinobu introduced a principle for guessing generalized clubs, denoted $\curlywedge^*(\kappa, S)$. They proved that it follows from $\diamondsuit^*(S)$, and showed how to derive a Souslin tree from it.

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König, Larson and Yoshinobu introduced a principle for guessing generalized clubs, denoted $\curlywedge^*(\kappa, S)$. They proved that it follows from $\diamondsuit^*(S)$, and showed how to derive a Souslin tree from it. Here is a weakening of their principle (that follows already from $\diamondsuit$):

Definition of $\curlywedge^-(\kappa, S)$, for $S \subseteq \lambda^+$

There exists a sequence $\langle \mathcal{C}_\alpha \mid \alpha \in S \rangle$ such that:

1. for all $\alpha \in S$, $\mathcal{C}_\alpha$ is a collection of $\leq \lambda$ many clubs in $[\alpha]^{<\kappa}$;
2. $\{\alpha \in S \mid \exists C \in \mathcal{C}_\alpha (C \subseteq \mathcal{D})\} \neq \emptyset$ for every club $\mathcal{D} \subseteq [\lambda^+]^{<\kappa}$.

Guessing of generalized clubs: $\curlywedge^-(\kappa, S)$

Let λ denote a regular uncountable cardinal.

Theorem 6

If $\lambda = 2^{<\lambda} < 2^\lambda = \lambda^+$ and $\curlywedge^-(\lambda, E^{\lambda^+}_\lambda)$ holds, then $\boxed{\diamondsuit}^{\{\lambda\}}_{\lambda,\lambda}$ is valid.

Guessing of generalized clubs: $\curlywedge^-(\kappa, S)$

Let λ denote a regular uncountable cardinal.

Theorem 6

If $\lambda = 2^{<\lambda} < 2^\lambda = \lambda^+$ and $\curlywedge^-(\lambda, E^{\lambda^+}_\lambda)$ holds, then $\boxed{\Diamond}^{\{\lambda\}}_{\lambda,\lambda}$ is valid.

Corollary (König-Larson-Yoshinobu, 2007)

If $\lambda = 2^{<\lambda} < 2^\lambda = \lambda^+$ and $\curlywedge^*(\lambda, E^{\lambda^+}_\lambda)$ holds, then there exists a $(<\lambda)$-complete λ^+-Souslin tree.

Schimmerling's question

Let λ denote a singular cardinal.

Question (Schimmerling, 2004)

Assuming GCH, for which μ, do $\square_{\lambda,<\mu}$ imply the existence of a λ^+-Souslin tree?

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Note

By Jensen, $\mu \geq 2$.
By Schimmerling, $\mu \geq \omega$.
By Theorem 4, $\mu \geq \text{cf}(\lambda)$.

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Note

By Jensen, $\mu \geq 2$.
By Schimmerling, $\mu \geq \omega$.
By Theorem 4, $\mu \geq \mathrm{cf}(\lambda)$.
Now, how about a larger μ? Specifically, will $\mu = \lambda^+$ work?

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Theorem 7

If $\lambda = 2^{<\lambda} < 2^\lambda = \lambda^+$ and there exists a non-reflecting stationary subset of $E^{\lambda^+}_{\neq \mathrm{cf}(\lambda)}$, then $\square_{\lambda,\lambda}$ implies $\boxtimes^{\Gamma}_{\lambda,\lambda}$.

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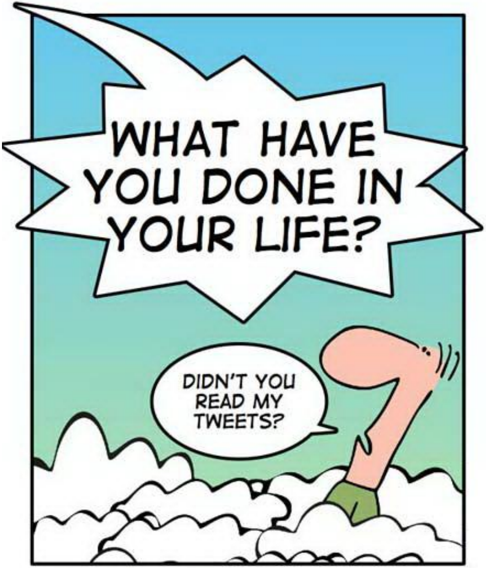
Partial answer (corollary)

$\mu = \lambda^+$, provided that $\exists$ non-reflecting stationary subset of $E^{\lambda^+}_{\neq \mathrm{cf}(\lambda)}$.

The extent of $\boxed{\Diamond}^{\Gamma}_{\lambda,<\mu}$



The extent of $\boxed{\diamondsuit}^{\Gamma}_{\lambda,<\mu}$

Theorem

In all the below cases, the following two are equivalent:

1. $\square_{\lambda,<\mu} + \mathsf{CH}_\lambda$
2. $\boxed{\diamondsuit}^{\Gamma}_{\lambda,<\mu}$

μ	λ	$\mathrm{cf}(\lambda)$	Remarks
$\mu = 2$	$\lambda \geq \aleph_1$	any	$\Gamma = \mathrm{Reg}(\lambda)$
$\mu \leq \mathrm{cf}(\lambda)$	$\lambda = \aleph_1$	any	$\Gamma = \mathrm{Reg}(\lambda)$, assuming CH
$\mu \leq \mathrm{cf}(\lambda)$	$\lambda > \aleph_1$	ctbl	$\Gamma = \{\theta\}$ and all large enough $\theta \in \mathrm{Reg}(\lambda)$
$\mu \leq \mathrm{cf}(\lambda)$	$\lambda > \aleph_1$	unctbl	Γ containing a final segment of $\mathrm{Reg}(\lambda)$
$\mu = \lambda^+$	$\lambda \geq \aleph_1$	any	some Γ, if: $2^{<\lambda} = \lambda$ & $\neg\,\mathrm{Refl}(E^{\lambda^+}_{\neq \mathrm{cf}(\lambda)})$
$\mu = \lambda^+$	sing.	any	some Γ, if: $2^{<\lambda} = \lambda$ & $\mathrm{SNR}(E^{\lambda^+}_{\mathrm{cf}(\lambda)})$

The definition of $\boxed{\Diamond}^{\Gamma}_{\lambda,<\mu}$

The definition of $\boxdiamond^{\Gamma}_{\lambda,<\mu}$

Definition

$\boxdiamond^{\Gamma}_{\lambda,<\mu}$ asserts the existence of two sequences, $\langle \mathcal{C}_\alpha \mid \alpha < \lambda^+ \rangle$ and $\langle \varphi_\theta \mid \theta \in \Gamma \rangle$, such that all of the following holds:

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- $\varphi_\theta : [\lambda^+]^{<\lambda} \to [\lambda^+]^{\leq\lambda}$ is a function, for all $\theta \in \Gamma$;

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    1. $\sup(\mathrm{acc}(C)) = \alpha$ for some $C \in \mathcal{C}_\alpha$;

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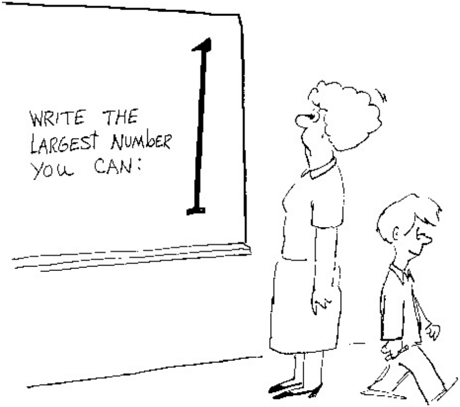
- Γ is a non-empty set of regular cardinals $< \lambda^+$;
- $\varphi_\theta : [\lambda^+]^{<\lambda} \rightarrow [\lambda^+]^{\leq\lambda}$ is a function, for all $\theta \in \Gamma$;
- for every subset $A \subseteq \lambda^+$, every club $D \subseteq \lambda^+$, and every $\theta \in \Gamma$, there exists some $\alpha \in E^{\lambda^+}_\theta$ such that:
  1. $\sup(\mathrm{acc}(C)) = \alpha$ for some $C \in \mathcal{C}_\alpha$;
  2. for every $C \in \mathcal{C}_\alpha$, either $\sup(\mathrm{acc}(C)) < \alpha$, or
  $$\sup\{\delta \in \mathrm{nacc}(\mathrm{acc}(C)) \cap D \mid \varphi_\theta(C \cap \delta) = A \cap \delta\} = \alpha.$$

Open Problems



Two questions

1. Assume $\square_{\lambda,\mathrm{cf}(\lambda)}$ + every stationary subset of λ^+ reflects. Can you find a $\square_{\lambda,\lambda}$-sequence $\langle \mathcal{C}_\alpha \mid \alpha < \lambda^+ \rangle$ such that for every club $D \subseteq \lambda^+$, there exists some $\alpha \in E^{\lambda^+}_{\neq \mathrm{cf}(\lambda)}$ with

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$$\sup(\mathrm{nacc}(C) \cap D) = \alpha \text{ for all } C \in \mathcal{C}_\alpha?$$

2. Let $\mathrm{Refl}(\lambda^+, \kappa)$ assert that for every stationary $S \subseteq E^{\lambda^+}_\kappa$, there exists some $\alpha \in E^{\lambda^+}_{>\kappa}$ for which $S \cap \alpha$ is stationary. Let $\mathrm{WRefl}(\lambda^+, \kappa)$ assert that for every $S \subseteq E^{\lambda^+}_\kappa$ and $f : S \to \lambda$, there exists some $\alpha \in E^{\lambda^+}_{>\kappa}$ such that $f \restriction C$ is non-injective for every club $C \subseteq \alpha$.
Question: Does $\mathrm{WRefl}(\lambda^+, \mathrm{cf}(\lambda))$ imply $\mathrm{Refl}(\lambda^+, \mathrm{cf}(\lambda))$?

Thank you!