Finitely approximable groups and actions

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• Suppose $G \curvearrowright X$ is an action of a countable group G on some discrete structure X. Under which conditions on G can this action be finitely approximated?

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Similarly, we have a related problem with a purely group theoretical motivation:

• When can a countable group G be finitely approximated?

Finitely approximable actions

To understand what we mean by finite approximation, we need a definition:

Suppose $G \curvearrowright X$ and $G \curvearrowright Y$ are actions of a group G on structures X and Y of the same category. For example, X and Y are just discrete sets, graphs, or metric spaces.

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An F-embedding of A into Y is a map $\pi\colon A \to Y$ such that

- $\pi\colon A \to Y$ is an isomorphic embedding,
- if $f \in F$ and $a, fa \in A$, then $\pi(fa) = f \cdot \pi(a)$.

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E.g., if X is just a discrete set, π being an isomorphic embedding just means that π is injective.

Definition

*An action $G \curvearrowright X$ of a countable group G on a structure X is said to be **finitely approximable** if for any finite subsets $A \subseteq X$ and $F \subseteq G$ there is an action $G \curvearrowright Y$ on a finite structure Y and an F-embedding $\pi \colon A \to Y$.*

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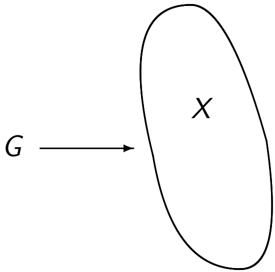

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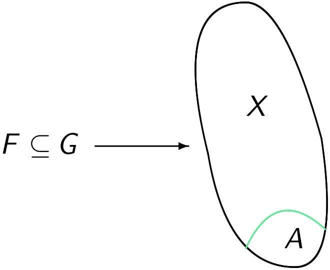

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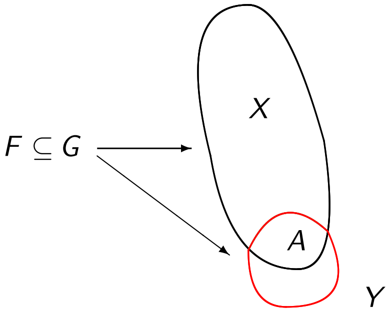

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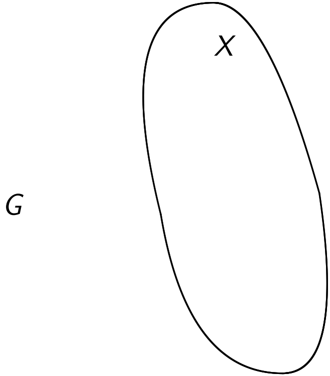

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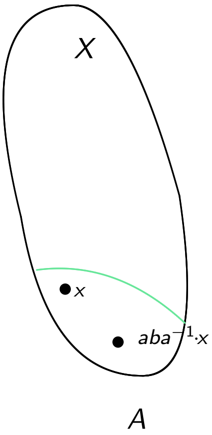

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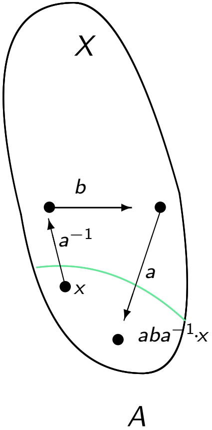

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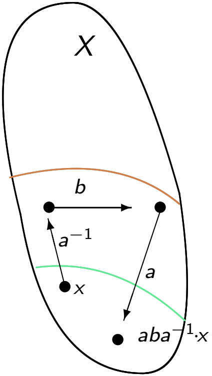

A

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A group G is residually finite if $\{1\}$ is a closed subgroup of G, i.e., if for all $g \neq 1$ there is a finite index subgroup $K \leqslant G$ with $g \notin K$.

For stronger properties of residual finiteness, one asks for more general subsets to be closed.

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Proposition

The following are equivalent for a countable group G.

- *G is subgroup separable,*
- *any action $G \curvearrowright X$ of G by permutations of a set X is finitely approximable.*

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M. Hall showed that free groups are subgroup separable.

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A group G has the Ribes-Zalesskiĭ property if any product

$$H_1 H_2 H_3 \cdots H_n = \{h_1 h_2 \cdots h_n \mid h_i \in H_i\}$$

of finitely generated subgroups $H_1, \ldots, H_n \leqslant G$ is closed in the profinite topology on G.

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Then with somewhat more work, one can prove the following.

Theorem

The following properties are equivalent for a countable group G.

- *G has the Ribes-Zalesskiĭ property,*
- *any action $G \curvearrowright X$ of G by isometries on a metric space X is finitely approximable.*

Moreover, one can control the metric type of the finite approximations.

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On the other hand, if one just wants to approximate actions of a group G by automorphisms of a graph, it suffices that products

$$H_1 H_2$$

of finitely generated subgroups $H_1, H_2 \leqslant G$ are closed.

Building structures from finite substructures

Definition

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Theorem (R. Fraïssé)

If $\mathcal{K}$ is a Fraïssé class, there is a unique countable structure $\mathbf{K}$*, which is ultrahomogeneous and whose finite substructures are exactly those of $\mathcal{K}$.*

We call $\mathbf{K}$ the *Fraïssé limit* of $\mathcal{K}$.

Example: The rational Urysohn metric space

If we let $\mathcal{K}$ be the class of finite metric spaces with rational distances, its limit $\mathbb{QU}$ is a countable rational metric space called the rational Urysohn metric space.

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The ultrahomogeneity of Fraïssé limits is simply that any finite partial automorphism extends to a full automorphism of the limit.

Thus, for example, any finite partial isometry of $\mathbb{QU}$, i.e., an isometry

$$f\colon A \xrightarrow{\cong} B,$$

where $A, B \subseteq \mathbb{QU}$ are finite subsets, extends to a full isometry

$$\tilde{f}\colon \mathbb{QU} \xrightarrow{\cong} \mathbb{QU}$$

of $\mathbb{QU}$ onto itself.

So the rational Urysohn space, $\mathbb{QU}$, functions as a universal model for rational metric spaces.

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E.g., using the universality and ultrahomogeneity of $\mathbb{QU}$ plus the fact that free groups have the Ribes-Zalesskiĭ property, one obtains

Theorem (S. Solecki, 2005)

For any finite (rational) metric space A there is a bigger finite (rational) metric space $Y \supseteq A$ such that any partial isometry of A extends to a full isometry of Y.

Automorphism groups

If X is a countable structure, we give $\mathrm{Aut}(X)$ the topology whose basic open sets are

$$\{g \in \mathrm{Aut}(X) \mid g(x_1) = y_1 \,\&\ldots\&\, g(x_n) = y_n\},$$

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So, for example, the topology on $\mathrm{Homeo}(2^{\mathbb{N}})$ and $\mathrm{Homeo}(2^{\mathbb{N}}, \mu)$ is just the compact-open topology, while the topology on $\mathrm{Aut}(\mathbb{QU}) = \mathrm{Isom}(\mathbb{QU})$ is finer than the pointwise convergence topology.

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$\mathrm{Aut}(X)$ is a totally disconnected, separable, completely metrisable group, but is in general not locally compact.

Definition

Suppose $\mathcal{K}$ is a Fraïssé class with limit $\mathbf{K}$. We say that $\mathcal{K}$ has the Hrushovski property if for any finite substructure $A \subseteq \mathbf{K}$ there is a bigger finite substructure $A \subseteq B \subseteq \mathbf{K}$ such that any partial automorphism of A extends to a full isomorphism of B.

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Note that not all Fraïssé classes have the Hrushovski property. For example, this fails for the class of finite linear orderings and finite Boolean algebras.

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However, using a theorem due to T. Coulbois stating that the Ribes-Zalesskiĭ property is stable under free products of groups, one can obtain the even stronger result

Theorem (S. Solecki)

$\mathrm{Isom}(\mathbb{QU})$ *has a locally finite, dense subgroup.*

Generic representations of f.g. groups

If Γ is a finitely generated group and $\mathcal{K}$ a Fraïssé class with limit $\mathbf{K}$, we let

$$\mathrm{Act}(\Gamma, \mathbf{K}) = \mathrm{Hom}(\Gamma, \mathrm{Aut}(\mathbf{K})) \subseteq \mathrm{Aut}(\mathbf{K})^\Gamma$$

be the space of all actions of Γ by automorphisms of $\mathbf{K}$ with the topology induced from $\mathrm{Aut}(\mathbf{K})^\Gamma$.

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So $\mathrm{Act}(\Gamma, \mathbf{K})$ is a closed subset of the completely metrisable space $\mathrm{Aut}(\mathbf{K})^{\Gamma}$.

We let $\mathrm{Aut}(\mathbf{K})$ act on $\mathrm{Act}(\Gamma, \mathbf{K})$ by conjugation of actions.

For example, if $\Gamma = \mathbb{F}_n$ is the free group on n generators, then

$$\mathrm{Act}(\Gamma, \mathbf{K}) = \mathrm{Aut}(\mathbf{K})^n,$$

while if $\Gamma = \mathbb{Z}^2$, then $\mathrm{Act}(\Gamma, \mathbf{K})$ is the set of commuting pairs in $\mathrm{Aut}(\mathbf{K})$.

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So a comeagre conjugacy class in $\mathrm{Aut}(\mathbf{K})$ is just a generic representation of $\mathbb{Z}$ in $\mathrm{Aut}(\mathbf{K})$.

Theorem

For any $n < \infty$, the free group $\mathbb{F}_n$ has a generic representation in the following automorphism groups

- $\mathrm{Aut}(\mathbf{R})$ *(Hrushovski, Hodges–Hodkinson–Lascar–Shelah)*
- $\mathrm{Isom}(\mathbb{QU})$ *(Solecki)*
- $\mathrm{Homeo}(2^{\mathbb{N}}, \mu)$ *(Kechris–C.R.)*

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This happens for example if Γ is a finitely generated Abelian group.

Theorem (I. Hodkinson, also J.K. Truss and K. Slutsky)

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But what about $\mathbb{F}_2$ in $\mathrm{Homeo}(2^{\mathbb{N}})$?

Structural information from generic representations

Using generic representations of $\mathbb{F}_n$, one can obtain strong information about a topological group connecting the topological and algebraic group structure.

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For example,

Theorem (A.S. Kechris–C.R.)

Let G be a complete metric group admitting a generic representation of $\mathbb{F}_n$ for every finite n. Then

- *every subgroup $H \leqslant G$ of countable index is open,*
- *any homomorphism $\pi\colon G \to H$ from G to a second countable topological group is continuous.*

Embedding the ring of finite adèles

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Briefly, define a topology on $\mathbb{Q}$ by letting

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if and only if $t_n \in \mathbb{Z}$ for all but finitely many n and also any integer k divides t_n for all but finitely many n.

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if and only if $t_n \in \mathbb{Z}$ for all but finitely many n and also any integer k divides t_n for all but finitely many n.

Alternatively, the topology on $\mathbb{Q}$ is given by the norm $\|\cdot\|$, where $\|0\| = 0$ and

$$\|s\| = 2^{-\min(n \mid \frac{s}{n} \notin \mathbb{Z})}.$$

The completion $\mathfrak{A}$ of $\mathbb{Q}$ under this topology is a zero-dimensional, locally compact ring (not a field) called the ring of finite adèles.

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We should think of $\mathfrak{A}$ as replacing the role of $\mathbb{R}$ in the category of totally disconnected groups. So continuous homomorphisms

$$X\colon \mathfrak{A} \to G$$

should be seen as the 1-parameter subgroups of a totally disconnected group G.

Abundance of 1-parameter subgroups

Theorem

The generic isometry $g \in \mathrm{Isom}(\mathbb{QU})$ *is in the image of a 1-parameter subgroup, in fact, there is a 1-parameter subgroup* $X \colon \mathfrak{A} \to \mathrm{Isom}(\mathbb{QU})$ *such that* $X(1) = g$*.*

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The generic measure-preserving homeomorphism $g \in \mathrm{Homeo}(2^{\mathbb{N}}, \mu)$ *is in the image of a 1-parameter subgroup* $X\colon \mathfrak{A} \to \mathrm{Homeo}(2^{\mathbb{N}}, \mu)$.

Abundance of 1-parameter subgroups

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For example, since for any homomorphism $X\colon \mathfrak{A} \to G$, we have

$$X\left(\frac{1}{2}\right)^2 = X\left(\frac{1}{2} + \frac{1}{2}\right) = X(1),$$

it follows that the generic isometry and generic measure-preserving homeomorphism have square roots.