On the strength of (failure of) square

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The main theorem

Theorem (S.)

Suppose that for some singular strong limit cardinal κ, $\square_\kappa$ fails. Then there is a nontame mouse. In particular, there is an inner model with a Woodin cardinal δ and a strong cardinal $\lambda < \delta$.

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A stronger lower bound

Remark

1. *One can show that in fact "$AD_{\mathbb{R}} + \Theta$ is regular" is a lower bound but that is a different story for some other time.*
2. *The proof is via core model induction and builds on Steel's proof that "$\neg\Box_\kappa \implies AD^{L(\mathbb{R})}$". (Recall that $AD^{L(\mathbb{R})}$ is equiconsistent with ω Woodins.)*

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What is the core model induction?

It is a technique for calibrating lower bounds of consistency strengths of set theoretic statements.

Typical applications of the core model induction

1. Forcing axioms: *PFA* and etc.
2. Combinatorial statements: $\neg\Box_\kappa$ where κ is a singular strong limit cardinal and etc.
3. Generic embeddings: generic embeddings given by precipitous ideals, dense ideals and etc.

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How does the core model induction work?

1. It can be viewed as a way of constructing models of determinacy while working in extensions of *ZFC*.
2. There is a collection of companion theorems that link the determinacy theories with large cardinal theories.
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Solovay sequence

First, recall that assuming AD,

$$\Theta = \sup\{\alpha : \text{there is a surjection } f : \mathbb{R} \to \alpha\}.$$

Then, assuming AD, the Solovay sequence is a closed sequence of ordinals $\langle \theta_\alpha : \alpha \leq \Omega \rangle$ defined by:

1. $\theta_0 = \sup\{\alpha :$ there is an ordinal definable surjection $f : \mathbb{R} \to \alpha\}$,
2. If $\theta_\alpha < \Theta$ then fixing $A \subseteq \mathbb{R}$ of Wadge rank θ_α, $\theta_{\alpha+1} = \sup\{\alpha :$ there is a surjection $f : \mathbb{R} \to \alpha$ such that f is ordinal definable from $A\}$,
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The hierarchy: Solovay hierarchy

$$AD^+ + \Theta = \theta_0 <_{con} AD^+ + \Theta = \theta_1 <_{con} ...AD^+ + \Theta = \theta_\omega <_{con}$$
$$...AD^+ + \Theta = \theta_{\omega_1} <_{con} AD^+ + \Theta = \theta_{\omega_1+1} <_{con} \cdots$$

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Connections to large cardinals

1. (Woodin, AD^+) $AD_{\mathbb{R}} \Leftrightarrow AD^+ +$ "$\Theta = \theta_\alpha$ for some limit α".
2. (Steel) $AD_{\mathbb{R}} \rightarrow$ there is a proper class model M of ZFC such that in M there is λ which is a limit of Woodin cardinals and $< \lambda$-strong cardinals.
3. (Woodin) If λ is a limit of Woodin cardinals and $< \lambda$-strong cardinals then the derived model at λ satisfies $AD_{\mathbb{R}}$.

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Theorem (Woodin)

$AD^+ + \Theta = \theta_1$ implies the existence of a nontame mouse. In particular, there is an inner model with a Woodin cardinal δ and a strong cardinal $\lambda < \delta$.

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Hence, to prove the theorem it is enough to construct a model of $AD^+ + \Theta = \theta_1$ from $\neg\Box_\kappa$ where κ is a singular strong limit cardinal.

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How do we construct models for these axioms?

1. Recall that the goal of CMI is to construct models for the axioms from the Solovay hierarchy. In our case, we want a model of $AD^+ + \Theta = \theta_1$.
2. With an apology to the experts, the model is essentially $L(\Gamma_{max}, \mathbb{R})$ where

$$\Gamma_{max} = \{A \subseteq \mathbb{R} : L(A, \mathbb{R}) \models AD^+\}.$$

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1. Core model induction is used to show that Γ_{max} has various closure properties.
2. Here we will concentrate on the following closure property.
3. Given a theory S from the Solovay hierarchy, is there $\Gamma \subseteq \Gamma_{max}$ such that $L(\Gamma, \mathbb{R}) \vDash S$ and $\Gamma = \mathcal{P}(\mathbb{R}) \cap L(\Gamma, \mathbb{R})$.
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1. Let κ be a singular strong limit cardinal such that $\neg\Box_\kappa$ and let $\mu > \mathrm{cf}(\kappa)$ be a regular cardinal such that $\mu^\omega = \mu$.
2. Fix $\lambda >> \kappa$ and let X be a submodel of V_λ such that letting N be the transitive collapse of X
   1. $\mu + 1 \subseteq \mathcal{N}$,
   2. $\mathcal{N}^\omega \subseteq \mathcal{N}$,
   3. $|\mathcal{N}| = \mu$.
3. Let $\pi : \mathcal{N} \to V_\lambda$.
4. We start working in $V[g]$ (backed up by V) where $g \subseteq Coll(\omega, \mu)$.
5. We can extend π to $\pi : \mathcal{N}[g] \to V_\lambda[g]$.

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2. In an earlier work, Steel has shown that $L(\Gamma_{max}, \mathbb{R}) \vDash AD^{+}$. Therefore, $L(\Gamma_{max}, \mathbb{R}) \vDash AD^{+} + \Theta = \theta_0$.
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The nature of A

1. The general idea is to produce a countable mouse $\mathcal{M}$ such that $\mathcal{M}$ has a "nice" iteration strategy Σ such that $Code(\Sigma) \notin \Gamma_{max}$.
2. Then use *CMI* to show that $L(Code(\Sigma), \mathbb{R}) \vDash AD^+$.
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The nature of $\mathcal{M}$

1. Let $\mathcal{H}^* = (V_\Theta)^{\mathrm{HOD}^{L(\Gamma_{max}, \mathbb{R})}}$ and let $\mathcal{P}^* = \pi^{-1}(\mathcal{H}^*)$. Let $\mathcal{H} = Lp_\omega(\mathcal{H}^*)$ and $\mathcal{P} = Lp_\omega(\mathcal{P}^*)$.
2. Under $AD^+ + MSC$, $\mathcal{H}$ is a mouse.
3. Hence, $\mathcal{P}$ is a mouse.
4. We then try to construct a strategy for $\mathcal{P}$. Let Σ be this strategy.
5. To show that $Code(\Sigma) \notin \Gamma_{max}$, it is enough to show that $\mathcal{H}$ is a Σ-iterate of $\mathcal{P}$.

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The final plan.

Construct a strategy Σ for $\mathcal{P}$ such that

1. HOD is a Σ-iterate of $\mathcal{P}$.
2. $L(Code(\Sigma), \mathbb{R}) \vDash AD^+$.

The construction of Σ

Diagram on the board.

Showing that $L(Code(\Sigma), \mathbb{R}) \vDash AD^+$

1. To show that $L(Code(\Sigma), \mathbb{R}) \vDash AD^+$ one needs to show that Σ has *branch condensation*.

Definition (Branch condensation)

An iteration strategy Σ has *branch condensation* if for any two stacks $\vec{\mathcal{T}}$ and $\vec{\mathcal{U}}$ on M_Σ and branch c of $\vec{\mathcal{U}}$ if

1. $\vec{\mathcal{T}}$ and $\vec{\mathcal{U}}$ are according to Σ, $lh(\vec{\mathcal{U}}) = \gamma + 1$ and $lh(\mathcal{U}_\gamma)$ is limit,
2. if $b = \Sigma(\vec{\mathcal{T}})$ then $i_b^{\vec{\mathcal{T}}}$ exists,
3. $i_c^{\vec{\mathcal{U}}}$-exists and for some $\pi : \mathcal{M}_c^{\vec{\mathcal{U}}} \to_{\Sigma_1} \mathcal{M}_b^{\vec{\mathcal{T}}}$,
$$i^{\vec{\mathcal{T}}} = \pi \circ i_c^{\vec{\mathcal{U}}}$$

then $c = \Sigma(\vec{\mathcal{U}})$.

1. The proof that our strategy Σ has branch condensation is rather technical.
2. Once it is done, however, *CMI* can be used to show that $L(Code(\Sigma), \mathbb{R}) \vDash AD^+$.

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What is needed to get more?

1. The proof of *Mouse Set Conjecture*.
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Mouse Capturing

Definition

The Mouse Capturing is the statement that for any two reals x and y, x is $OD(y)$ iff there is a mouse $\mathcal{M}$ over y such that $x \in \mathcal{M}$.

The Mouse Set Conjecture

Conjecture (Steel and Woodin)

Assume AD^+ and that there is no inner model with a superstrong cardinal. Then Mouse Capturing holds.

Instances of the Mouse Capturing

Theorem

1. *(Kleene)* $x \in \Delta^1_1(y) \leftrightarrow x \in L_{\omega_1^{ck}(y)}[y]$.
2. *(Shoenfield)* x *is* $\Delta^1_2(y)$ *in a countable ordinal iff* $x \in L[y]$.
3. *etc.*

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3. *etc.*

A partial result

Theorem (S.)

Assume AD^+ and there is no inner model containing the reals and satisfying $AD_{\mathbb{R}}$ + "Θ is regular". Then Mouse Capturing holds.

How are hods computed?

Assume Mouse Capturing and work under AD^+. As a first step, notice that if $x \in \mathrm{HOD}$ then x is in a mouse. So $\mathbb{R}^{\mathrm{HOD}}$ is a set of reals of a mouse. We just generalize this but it is much harder.
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The hod theorems

Theorem (S.)

HOD *of the minimal model of* $AD_{\mathbb{R}} +$ *"*Θ *is regular" is a hod premouse.*

This much is enough to carry out the general outline and get "$AD_{\mathbb{R}} + \Theta$ is regular" as a lower bound for $\neg\Box_{\kappa}$.

The definition of a hod mouse is motivated by the following theorem of Woodin.

Theorem (Woodin)

Assume AD$^+$. For every α, if $\theta_{\alpha+1}$ exists then it is a Woodin cardinal in HOD.

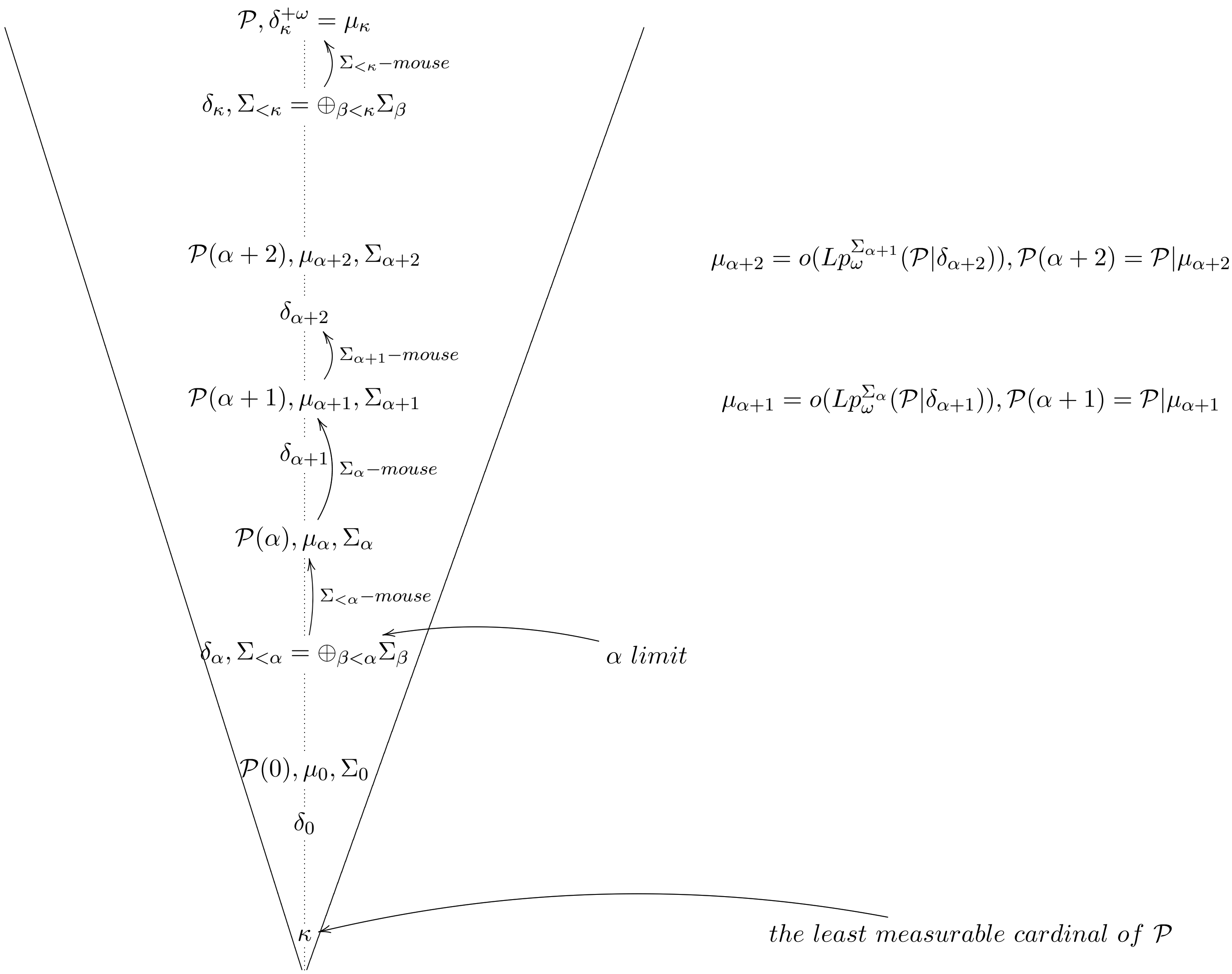

Figure 0.1: Hod premouse with $\mathcal{P} \vDash$ "$\lambda^{\mathcal{P}}$ =the least measurable cardinal κ".