Generalisations of scattered orders

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Density

Definition

- An order P is *dense* iff for every $a < b$ in P, there is $c \in P$ such that $a < c < b$.
- An order P is κ-*dense* iff for every $a < b$ in P, there is $C \subset P$ with $|C| \geq \kappa$ such that $a < C < b$.
- An order P is κ-*saturated* iff for every $A < B$ subsets of P with $|A|, |B| < \kappa$, there is $c \in P$ such that $A < c < B$.

If $\kappa^{<\kappa} = \kappa$ then let $\mathbb{Q}(\kappa)$ be the κ-saturated linear order of size κ.
This was defined by Hausdorff in 1908.

Scattered

Definition

- An order is *scattered* iff it does not embed the rationals.
- An order is κ-*scattered* iff it does not embed any κ-dense order.
- An order is $\mathbb{Q}(\kappa)$-*scattered* iff it does not embed $\mathbb{Q}(\kappa)$.

Constructing hierarchies of orders

- Stage 0: Start with a base class of “simple” orders (e.g. well orders)
- Stage $\alpha + 1$: Take the closure of orders at stage α under “natural” order operations (e.g. lexicographic sums and inversions)
- Stage δ limit: Take the union of orders at all stages $\beta < \delta$

The example was proved by Hausdorff to be exactly the class of scattered linear orders.

Classification of κ-scattered linear orders

Theorem

The κ-scattered linear orders form a class which is exactly the closure of the class of all well-orders and linear orders of size $< \kappa$ under inversions and lexicographic sums.

The theorem was originally proved by Hausdorff for $\kappa = \aleph_0$.

Applications of the classification

Corollary

- *Restricting the closure as in Theorem 1 to linear orders of size λ for λ regular gives a characterisation κ-scattered linear orders of size λ.*
- *If L is a linear order of size κ and for all $\alpha < \kappa^+$ there is an embedding $\alpha \hookrightarrow L$ then L embeds a κ-dense order.*
- *The $\aleph_1$-scattered linear orders are bqo under embeddability.*

The final statement relies on a theorem of Laver that the σ-scattered linear orders are bqo under embeddability.

Generalisation of κ-dense to posets

Definition

- A poset is κ-fat iff it is not an antichain and for every $a < b$ in P, the interval (a, b) has cardinality at least κ.
- A poset is FAC iff it has no infinite antichain.

Theorem

If P is an FAC κ-fat poset, then P embeds a κ-dense linear order.

The necessity of FAC to embed a κ-dense LO

Theorem

There is an $\aleph_1$-fat poset with no uncountable chain or uncountable antichain.

Generalising scattered to posets and their augmentations

Definition

- A poset is κ-*scattered* iff it does not embed any κ-dense order.
- A poset is $\mathbb{Q}(\kappa)$-*scattered* iff it does not embed $\mathbb{Q}(\kappa)$.

Theorem (Džamonja, T.)

Any augmentation of an FAC κ-scattered ($\mathbb{Q}(\kappa)$-scattered) poset is κ-scattered ($\mathbb{Q}(\kappa)$-scattered).

Szpilrajn showed that every poset can be augmented to a linear order.

The necessity of FAC to preserve κ-scattered under augmentation

Theorem

There is an uncountable poset Q such that

- *Q has no uncountable chains or antichains*
- *Q has an $\aleph_1$-dense augmentation to a linear order with no decreasing $\aleph_1$-sequence.*

Classification of κ-scattered FAC posets.

A wqo poset is a well-founded poset which has no infinite antichain (FAC).

Theorem

Let $\mathscr{B}_\kappa$ be the class of posets P such that P is either a wqo poset, the reverse of a wqo poset or a linear ordering of size $< \kappa$.

Let $\mathscr{P}_\kappa$ be the closure of $\mathscr{B}_\kappa$ under augmentations and lexicographic sums with index set in $\mathscr{B}_\kappa$.

Then $\mathscr{P}_\kappa$ is the class of κ-scattered FAC posets.

Abraham and Bonnet proved this result for $\kappa = \aleph_0$.

Structure theorem for countable FAC posets

Corollary

Let $\mathscr{B}$ be the class of countable posets P such that P is either a wqo poset, the reverse of a wqo poset or a linear order.

Let $\mathscr{C}$ be the closure of $\mathscr{B}$ under augmentations and lexicographic sums with index set in $\mathscr{B}$.

Then $\mathscr{C}$ is the class of countable FAC posets.

Classification of $\mathbb{Q}(\kappa)$-scattered FAC posets.

Theorem

Let $\mathscr{B}_\kappa$ be the class of posets P such that P is either a wqo poset, the reverse of a wqo poset or a $\mathbb{Q}(\kappa)$-scattered linear order.

Let $\mathscr{P}_\kappa$ be the closure of $\mathscr{B}_\kappa$ under augmentations and lexicographic sums with index set in $\mathscr{B}_\kappa$.

Then $\mathscr{P}_\kappa$ is the class of $\mathbb{Q}(\kappa)$-scattered FAC posets.

The problem with classifying $\mathbb{Q}(\aleph_1)$-scattered orders

Observation

- *(Goldstern) If W is an outer model of V such that there is a real in $W \setminus V$ then $\mathbb{Q}(\aleph_1)^V$ is not $\aleph_1$-saturated in W.*
- *If W is an outer model of V such that these models have the same reals then $\mathbb{Q}(\aleph_1)^V$ is $\aleph_1$-saturated in W.*

Theorem

There is a forcing extension $\mathbb{P}$ of V which preserves reals and a linear order L such that in V there is no embedding $\mathbb{Q}(\aleph_1) \hookrightarrow L$, but such an embedding exists in $V^{\mathbb{P}}$.

Future directions

- Are there set-theoretic assumptions we can make, under which there is a classification of $\mathbb{Q}(\aleph_1)$-scattered linear orders?
- What other types of orders can be classified using this Hausdorff-type constructive hierarchy?
- What else can we infer from the classification of κ-scattered orders?

Thanks!

Thank you for your attention!