On the consistency strength of the proper forcing axiom

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MAIN RESULT

Definition 1 $\{(P_\alpha, \dot{Q}_\beta) : \alpha \leq \kappa, \beta < \kappa\}$ *is a standard iteration of length* κ *if:*

$|P_\alpha| < \kappa$ *for all* $\alpha < \kappa$*,*

P_α *is a direct limit for stationary many* $\alpha < \kappa$*.*

Theorem 2 *Assume* PFA *is proved consistent by means of a standard iteration* $\{(P_\alpha, \dot{Q}_\beta) : \alpha \leq \kappa, \beta < \kappa\}$ *such that:*

- P_κ is proper,
- κ *is* $\aleph_2$ *in the generic extension.*

Then κ *is* supercompact *in the ground model.*

There is an essential contribution by Magidor in the proof of this result.

Without Magidor's contribution this is the optimal result:

Theorem 3 *Assume* PFA *is proved consistent by means of a standard iteration of length* κ *such that:*

- κ *is* $\aleph_2$ *in the generic extension.*

Then κ *is at least* strongly compact *in the ground model.*

Hiroshi Sakai has shown that there is a huge obstruction in order to get the optimal result without the assumption that the iteration is proper.

HOW TO GET TO THE MAIN RESULT:

STEP 1: Combinatorial characterization of supercompactness and strong compactness.

There are combinatorial properties ISP(κ) and SP(κ) such that ISP(κ) implies SP(κ) and:

- κ is *supercompact* iff κ is inaccessible and ISP(κ) holds (Magidor 1974).

- κ is *strongly compact* iff κ is inaccessible and SP(κ) holds (Jech 1973).

- ISP(κ) can hold even for κ a successor of a regular uncountable cardinal (Weiss 2008) !!!

STEP 2: PFA **implies that "**$\aleph_2$ **is supercompact" i.e** ISP($\aleph_2$) **holds(WEISS, viale 2009)**

STEP 3: In many circumstances ISP(κ) **and** SP(κ) **can be relativized to inner models**

For what concerns SP(κ):

Theorem 4 (Viale, 2010) *If W is a generic extension of V by a standard iteration of length κ and:*

- SP(κ) *holds in W,*

- *κ is inaccessible in V,*

Then κ is strongly compact in V.

For what concerns ISP(κ):

Theorem 5 (MAGIDOR, viale, 2010)

If W is a generic extension of V by a standard iteration $\{(P_\alpha, \dot{Q}_\beta) : \alpha \leq \kappa, \beta < \kappa\}$ and:

- P_κ is proper,
- ISP(κ) *holds in W,*
- *κ is inaccessible in V,*

Then κ is supercompact *in V.*

Some words on step 1

Let $j : V \to M \subseteq V$ be elementary with M a transitive class and assume:

- $j(\kappa) > \lambda$,
- $V_\lambda \in M$,
- eventually $j[V_\lambda] \in M$.

If 1 and 2 hold, j witnesses κ is at least V_λ-strong.

If 1,2,3 hold, j witnesses κ is at least V_λ-supercompact:

Let $\mathcal{U}$ be the normal measure on $[V_\lambda]^{<\kappa}$ given by

$$A \in \mathcal{U} \text{ iff } j[V_\lambda] \in j(A)$$

Notice $j[V_\lambda] = N \prec M_{j(\lambda)}$.

N has the following crucial property:

Fact 1 *For every $X \in N$ and every $d \in P(X)^M$ (d may not be in M), there is $z \in N$ such that $z \cap N = d \cap N$.*

Proof: N is isomorphic to V_λ,
thus we can find $d' \in V_\lambda$ such that $j[d'] = d \cap N$.
Then $j(d') = z$ is as required. □

Magidor has charcterized supercompactness using models with this "*guessing*" property.

Definition 6 *Given some $N \prec V_\lambda$ and some $d \subseteq X$ for some $X \in N$, we say that d is N-*guessed *if $d \cap N = z \cap N$ for some $z \in N$.*

Which sets d can be N-guessed?

It depends on the structure N and on the choice of the sets d.

In the terminology of the previous slide:

$M \models$*Every $d \subseteq X$ is $j[V_\lambda]$-guessed for any $X \in j[V_\lambda]$.*

On the other hand

Assume $M = \bigcup\{M_\alpha : \alpha < \omega_1\} \prec H(\aleph_3)$ is internally approachable of size $\aleph_1$.

Let $C = \{M_\alpha \cap \aleph_2 : \alpha < \omega_1\}$. Then C cannot be guessed. Why?

Otherwise C would be guessed by a $D \in M$ such that $D \cap M$ is unbounded in $\aleph_2 \cap M$.

Thus M models D is an unbounded subset of $\aleph_2$.

Thus $\omega_1 = \text{otp}(C) = \text{otp}(D \cap M) = M \cap \aleph_2 > \omega_1$.

Notice however that all the initial segments of C are in M.....

CONCLUSION:

An internally approachable model M of size $\aleph_1$ contains a subset $C \subseteq M \cap \aleph_2$ such that:

1. $C \cap X \in M$ for all countable $X \in M$,

2. C is not guessed, i.e. $C = C \cap M \neq E \cap M$ for all $E \in M$.

BUT:

If $M \prec H(\theta)$ has size $\aleph_1$, then for any set C which is M-guessed, item 1 above holds.

QUESTION: Can there be an $M \prec H(\theta)$ of size $\aleph_1$ such that item 1 above is a sufficient condition for a set to be M-guessed?

Let R be a suitable initial segment of the universe V ($R = H(\theta)^V$ or $R = V_\lambda$). What matters is:

- R satisfies enough axioms of ZFC,
- R is a transitive set,
- $P(X)^V \subseteq R$ for all $X \in R$.

Let $N \prec R$ be a substructure.

Define

$$\kappa_N = \min\{\alpha : N \cap \alpha + 1 \text{ is not an ordinal}\}$$

$$\overline{\kappa}_N = \sup\{|\alpha| : N \cap \alpha \text{ is an ordinal}\}$$

It is easy to check that κ_N and $\overline{\kappa}_N$ are cardinals in V.

Two illuminating examples to compute κ_N and $\overline{\kappa}_N$:

1. If $N \prec H(\theta)$, $|N| = \omega_1$ and $\omega_1 \subseteq N$, then $\overline{\kappa}_N = \omega_1$ and is in N and $\kappa_N = \aleph_2$ is also in N.

2. According to the previous slides, $\overline{\kappa}_{j[V_\lambda]} = \kappa \notin j[V_\lambda]$ is an inaccessible cardinal while $\kappa_{j[V_\lambda]} = j(\kappa) \in j[V_\lambda]$.

Definition 7 *(Weiss, Viale) Let $N \prec H(\theta)$ $(N \prec V_\lambda)$ and $X \in N$.*

$d \subseteq X$ is an N-slender subset of X if $d \cap Z \in N$ for all $Z \in [N]^{<|\overline{\kappa_N}|}$.

Remark 8 *Assume $\overline{\kappa_N}$ is inaccessible (it is enough that $\overline{\kappa_N}^{<\overline{\kappa_N}} = \overline{\kappa_N}$).*
Then for every $X \in N$ every $d \subseteq X$ is N-slender.

Proof: Notice that if $Z \in N$ has size less than $\overline{\kappa}_N$, $P(Z) \subseteq N$, thus $d \cap Z \in N$ for any set d. $\square$

Definition 9 *Let $N \prec H(\theta)$ ($N \prec V_\lambda$) and $X \in N$.*

N is an X-guessing model if every N-slender subset of X is N-guessed.

N is a guessing model if it is X-guessing for all $X \in N$.

Example:

If $j : V \to M$ is elementary and such that $V_\lambda \subseteq M$,

$N = j[V_\lambda] \prec M_{j(\lambda)}$ is a guessing model with respect to the universe M (even if N might not be in M).

j witnesses the V_λ-supercompactness of κ in V if $N \in M$.

Definition 10 *Given a cardinal κ,* ISP(κ) *holds if:*

For all $\theta > \kappa$ there are stationarily many N in $[H(\theta)]^{<\kappa}$ such that:

- *N is a guessing models,*

- *$\kappa_N = \kappa$.*

Theorem 11 (Magidor, 1974) *κ is supercompact iff it is inaccessible and* ISP(κ) *holds.*

STEP 2:

Theorem 12 (WEISS, viale, 2009) PFA *implies* ISP($\aleph_2$) *holds.*

Proof: Use PFA to find a model $M \prec H(\theta)$ of size $\aleph_1$ which has an M-generic filter for a variation of the poset to show that the approachability property fails at $\aleph_2$. $\square$

STEP 3: How to relativize ISP(κ) to inner models.

Definition 13 (Hamkins?, Laver?) *Let $V \subseteq W$ be transitive models of* ZFC.

The pair (V, W) has the κ-covering property if:

Every $X \in [Ord]^{<\kappa} \cap W$ is covered by some $Y \in [Ord]^{<\kappa} \cap V$.

The pair (V, W) has the κ-approximation property if:

For every set of ordinals $X \in W$ such that:

$$X \cap Z \in V \text{ for all } Z \in [Ord]^{<\kappa} \cap V,$$

we actually have that $X \in V$.

Theorem 14 (Laver, 1999) *Assume W is a generic extension of V.*
Then for some κ the pair (V, W) has the κ-covering property and the κ-approximation property.

Theorem 15 (Viale, 2010) *Assume $W = V[G]$ where G is a P_κ-generic filter for a standard iteration $\{(P_\alpha, \dot{Q}_\beta) : \alpha \leq \kappa, \beta < \kappa\}$.*
Then the pair (V, W) has the κ-covering property and the κ-approximation property.

Theorem 16 *Assume*

- V *models that* $\{(P_\alpha, \dot{Q}_\beta) : \alpha \leq \kappa, \beta < \kappa\}$ *is a standard iteration,*
- $V \models \kappa$ *is inaccessible,*
- G *is* V*-generic over* P_κ*,*
- P_κ *is proper,*
- $W = V[G] \models \mathsf{ISP}(\kappa)$.

Then V *models that* κ *is supercompact.*

Notice that in the previous theorem κ may not be anymore inaccessible in W.

If I skip the proof remember me to go to the last slide!

Proof: For the sake of simplicity assume that:

$\lambda > \kappa$ is inaccessible in W,

Fix in V a bijection f between λ and $V_\lambda = H(\lambda)^V$.

Fix also in V a partition $\mathcal{S} = (S_\alpha : \alpha < \lambda)$ of the ordinals of countable cofinality below λ.

Remark that $\mathcal{S}$ remains a partition of stationary sets in W.

Take in W, a guessing model $M \prec H(\lambda^+)^W$ such that

- $\kappa_M = \kappa$ and $|M| < \kappa$.
- $P_\kappa, f, \mathcal{S}........ \in M$.

The two crucial observations are the following:

Fact 2 $M \cap V \prec H(\lambda^+)^V$ *is a guessing model with respect to* V.

Fact 3 $M \cap V_\lambda \in V$.

Then we can conclude using Magidor's characterization of supercompactness in V.

I will prove both facts in some detail.

The first fact relies on the assumption that $W \models \mathsf{ISP}(\kappa)$.

The second relies on the assumption that W is an extension of V by a proper forcing.

First of all standard arguments using the fact that $M \prec H(\lambda^+)^W$ and $M \cap \kappa$ is an ordinal show that:

1. $M = M \cap V[G \cap M]$ (Use that P_κ has the κ-CC),

2. $M \cap V \prec H(\lambda^+)^V$,

3. $P(Z)^V \subseteq M \cap V$ for all $Z \in M \cap V$ whose size in V is less than $M \cap \kappa$.

   (Use that κ is inaccessible in V, the elementarity of $M \cap V$ and the fact that $M \cap \kappa$ is an ordinal),

4. Every $Z \in M \cap [\lambda]^{<\kappa}$ is covered by some $Y \in M \cap V \cap [\lambda]^{<\kappa}$

   (Use the κ-covering property of the pair (V, W)).

Proof of the first fact:

First step: we want to show that any $d \in H(\lambda^+)^V$ is an M-slender subset of M.

It is enough to check that this is the case for any $d \in P(\lambda)^V$ since any set in $H(\lambda^+)^V$ can be coded by a subset of λ.

Pick $Z \in M \cap V \cap [\lambda]^{<\kappa}$,
then by the third item of the previous slide $d \cap Z \in P(Z)^V \subseteq M \cap V$.

Thus for all $Z \in M \cap V \cap [\lambda]^{<\kappa}$, $d \cap Z \in M$.

Now if $Y \in (M \setminus V) \cap [\lambda]^{<\kappa}$,
M models that there is $Z \in V \cap [\lambda]^{<\kappa}$ such that $Y \subseteq Z$.

Thus $d \cap Y = d \cap Y \cap Z$. But $d \cap Z \in M$, so $d \cap Y$ is also in M.

Thus d is an M-slender subset of λ,
since $d \cap Y \in M$ for all $Y \in [\lambda]^{<\kappa}$.

Since M is a guessing model,
$d \cap M = e \cap M$ for some $e \in M$.

Now $e \cap Z = d \cap Z \in V$ for all $Z \in [\lambda]^{<\kappa_M} \cap V \cap M$. Thus

$$M \models e \cap Z \in V \text{ for all } Z \in V \cap [\lambda]^{<\kappa}$$

By elementarity of M:

$$W \models e \cap Z \in V \text{ for all } Z \in V \cap [\lambda]^{<\kappa}.$$

By the κ-approximation property of the pair (V, W) we have that $e \in V$.

In conclusion for every $d \in V$ there is $e \in M \cap V$ such that $d \cap M = e \cap M$, i.e. $M \cap V \prec H(\lambda^+)^V$ is a guessing model.

This proves the first fact. $\square$

Proof of the second fact:

Let $\delta = \text{sup}(M \cap \lambda)$.

This is the key observation:

Claim 17 *For every $S \in P(\lambda)^V \cap M$ set of limit ordinals of countable cofinality, we have that in V*

$V \models S$ is stationary *iff* $V \models S$ reflects on δ.

Proof of the claim

First of all with some work it can be seen that for any guessing model $M \prec H(\lambda^+)^W$, $M \cap \lambda$ is closed under countable supremum, i.e.:

If $\{\alpha_n : n \in \omega\} \subseteq M \cap \lambda$, then $\alpha = \sup\{\alpha_n : n \in \omega\} \in M$.

Pick $S \in M \cap V$ set of ordinals of countable cofinality.

First assume:

$V \models S$ *reflects on* δ.

Pick $C \in M$ club subset of λ,
then $C \cap M$ is a countably closed subset of δ so it meets S.

Thus $M \models S$ *is stationary* .

So S is stationary in W and a fortiori in V.

Assume on the other hand that $V \models S$ *is stationary*, we want to show that S reflects on δ.

By the κ-CC of P_κ every $C \in W$ club subset of λ is contained in a club in V.

Thus if $S \in M \cap V$:

$V \models S$ *is stationary*

iff

$W \models S$ *is stationary*

iff

$M \models S$ *is stationary*.

Now pick $C \in V$ club subset of δ, we want to show that $C \cap S$ is non-empty.

We just saw C is an M-slender subset of λ.

Since M is a guessing model $C \cap M = E \cap M$ for some $E \in M$.

Since C is a club we can easily argue that:

$$M \models E \text{ is closed under countable suprema.}$$

Since

$$M \models S \text{ is stationary,}$$

we get that

$$M \models S \cap E \neq \emptyset.$$

Thus $S \cap C \neq \emptyset$.

Since $C \in V$ is an arbitrary club subset of δ,

$$V \models S \text{ reflects on } \delta.$$

The claim is proved $\square$

We can conclude the proof of the second fact:

Since P_κ is proper, we get that for any $S \in P(\delta)^V$ set of limit ordinals of countable cofinality,

$V \models S$ *is a stationary subset of* δ iff $W \models S$ *is a stationary subset of* δ.

Now observe that
$M \cap V_\lambda = f[M \cap \lambda]$,
where $f \in M \cap V$ is the bijection we chose between λ and V_λ.

Observe also that $(S_\alpha : \alpha < \lambda) \in M$.

Thus for any $\alpha < \lambda$:

If $\alpha \in M$, then $S_\alpha \in M$ and
$M \models S_\alpha$ *is stationary in* λ
so
$V \models S_\alpha$ *reflects on* λ.

On the other hand

$V \models S_\alpha$ *reflects on* δ iff $W \models S_\alpha$ *reflects on* δ

because W is an extension of V by a proper forcing.

Since $M \cap \lambda$, is a subset of δ closed under countable suprema,
$M \cap S_\alpha \neq \emptyset$
and since $(S_\alpha : \alpha < \lambda) \in M$, we get that $\alpha \in M$.

In conclusion:

$\alpha \in M$ iff $V \models S_\alpha$ *reflects on* δ.

Thus

$$M \cap \lambda = \{\alpha : V \models S_\alpha \text{ reflects on } \delta\} \in V$$

This means that $M \cap V = f[M \cap \lambda] \in V$ and proves the second fact. □

What happens if P_κ is not proper?

Theorem 18 (Sakai 2010) *Assume $P_\kappa \in V$ is the final poset of the standard semiproper iteration to force* MM *of length a supercompact cardinal κ.*

Assume there are class many Woodin cardinals.

Let G be V-generic for P_κ and $W = V[G]$. Then for every θ there are stationarily many $M \in [H(\theta)^W]^{\aleph_1}$ which are guessing models and such that $M \cap V$ is a guessing model but is not in V.