Workshop on
Set theory and C*-algebras
January 23 to January 27, 2012
American Institute of Mathematics, Palo Alto, California
organized by
Ilijas Farah (York) and David Kerr (Texas A&M)
Interested students and postdocs can apply online:
http://www.aimath.org/
Some open problems on set theory and C*-algebras

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3rd(!) European Set Theory Conference, Edinburgh, June 2011
C*-algebras

$H$: a complex Hilbert space
C*-algebras

$H$: a complex Hilbert space

$\mathcal{B}(H)$: the algebra of bounded linear operators on $H$
C*-algebras

$H$: a complex Hilbert space

$(\mathcal{B}(H), +, \cdot, \cdot^*, \| \cdot \|)$: the algebra of bounded linear operators on $H$
**C*-algebras**

\[ H: \text{ a complex Hilbert space} \]
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**Definition**

A (concrete) \textit{C*-algebra} is a norm-closed subalgebra of \( \mathcal{B}(H) \).
C*-algebras

\[ H: \text{ a complex Hilbert space} \]
\[ (B(H), +, \cdot, *, \| \cdot \|): \text{ the algebra of bounded linear operators on } H \]

Definition
A (concrete) \textit{C*-algebra} is a norm-closed subalgebra of \( B(H) \).

Theorem (Gelfand–Naimark–Segal, 1942)
A Banach algebra with involution \( A \) is isomorphic to a concrete C*-algebra if and only if

\[ \| aa^* \| = \| a \|^2 \]

for all \( a \in A \).
Examples

(1) $B(H)$, $M_n(C)$.

(2) If $X$ is a compact Hausdorff space, $C(X)$.

$C(X) \sim \Rightarrow C(T) \Leftrightarrow X \sim \Rightarrow Y$.

(3) $K(H)$ - the algebra of compact operators on $H$. 

Examples

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(1) $\mathcal{B}(H), M_n(\mathbb{C})$. 

$K(\mathcal{H})$ - the algebra of compact operators on $H$. 
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(2) If $X$ is a compact Hausdorff space, $C(X)$.

\[ C(X) \cong C(T) \iff X \cong Y. \]

(3) $\mathcal{K}(H)$ - the algebra of compact operators on $H$. 
1. Classification problems

*UHF algebras* are direct limits of full matrix algebras, $M_n(\mathbb{C})$. 

Unital, separable UHF algebras are classified by reals. 

AF algebras are direct limits of finite-dimensional C*-algebras. Theorem (Elliott, 1975) 

Separable unital AF algebras are classified by the ordered (countable, abelian) group $(K_0(A), K_0(A) + 1)$. (Actually, every isomorphism between groups lifts to an isomorphism between algebras.)
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Elliott program

Many other subclasses of nuclear, separable, simple, unital C*-algebras, were classified until 2003 by the Elliott invariant,

$$\text{Ell}(A) : (\left(K_0(A), K_0(A)^+, 1\right), K_1(A)).$$
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Counterexamples

Jiang–Su, 1999: \[ \text{Ell}(\mathbb{Z}) = \text{Ell}(\mathbb{C}), \text{Ell}(A \otimes \mathbb{Z}) = \text{Ell}(A). \]

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The basic concept of abstract classification

Definition
If \((X, E)\) and \((Y, F)\) are analytic equivalence relations on standard Borel spaces, \(E\) is \textit{Borel-reducible} to \(F\), in symbols

\[
E \leq_B F,
\]

if there is a Borel-measurable map \(f : X \to Y\) such that

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x E y \iff f(x) E f(y).
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Theorem (F.–Toms–Törnquist, 2010)
This definition applies to all categories occurring in the Elliott program.
isomorphism of Banach spaces

C*-part by
Farah–Toms–Törnquist

biembeddability of AF

$E_{K_\sigma}$

orbit equivalence relations

Cuntz semigroups

$E_{G_\infty}$

simple nuclear

Elliot invariant

isometry of reflexive Banach spaces

compact convex*

simple AI

Choquet simplex

abelian $C^*$-algebras

compact metric

countable structures

$E_{1}$

smooth

$E_{0}$

biembeddability of UHF

UHF
Classification problems

Question

Is the isomorphism of all separable, unital C*-algebras $\leq_B$ orbit equivalence relation of a Polish group action?

What about other classes of C*-algebras?

Question

Is there a set of (nuclear, simple, separable, unital) C*-algebras with the same Elliott invariant on which the isomorphism is not classifiable by countable structures?

(Toms, 2008: There is such a set of size $c$.)

Question

Is the bi-embeddability relation for nuclear simple separable C*-algebras a complete analytic equivalence relation?

What about bi-embeddability of AF algebras?
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In the Kechris’s space of separable C*-algebras

Problem

*Compute the complexity of different classes of C*-algebras.* . . .

(Done by Kechris in some cases.)
In the Kechris’s space of separable C*-algebras

Problem
Compute the complexity of different classes of C*-algebras. . .
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Problem
. . . then use this to prove e.g., that

\[ \text{bootstrap class } \neq \text{ nuclear}. \]
More open-ended problems

Problem

*Develop set-theoretic framework for Elliott’s functorial classification.*
More open-ended problems

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*Develop set-theoretic framework for Elliott’s functorial classification.*

Problem
*Apply K-theory to set theory.*
2. My favourite

A representation $\pi : A \rightarrow \mathcal{B}(H)$ is irreducible (irrep) if $\pi[A]$ is dense in $\mathcal{B}(H)$.
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$\pi_1 \sim \pi_2$ iff there is an isometry $u$ such that:

\[
\begin{align*}
A \xrightarrow{\pi_1} \mathcal{B}(H_1) & \quad H_1 & (\text{Ad } u)a = uau^* \\
\pi_2 & \xrightarrow{\text{Ad } u} \mathcal{B}(H_2) & H_2
\end{align*}
\]

Theorem (Naimark) If $A = K(H)$ then all irreps of $A$ are equivalent.

Question (Naimark) If all irreps of $A$ are equivalent, is $A \sim K(H)$ for some $H$?
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& \mathcal{B}(H_2) \overset{u}{\longrightarrow} H_2
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**Theorem (Naimark)**

*If \( A = \mathcal{K}(H) \) then all irreps of \( A \) are equivalent.*

**Question (Naimark)**

*If all irreps of \( A \) are equivalent, is \( A \cong \mathcal{K}(H) \) for some \( H \)?
What is known about Naimark’s problem?

Theorem (Akemann–Weaver, 2002)

◊ **implies there is a unital, nonseparable** $A$ **with all irreps equivalent.**
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$A$ unital + $\aleph_1$-dimensional $\Rightarrow A \not\sim \mathcal{K}(H)$. 

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Question

Does CH imply there is a counterexample to Naimark’s Problem (NP)?
What is known about Naimark’s problem?
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If we add supercompact many Cohen/random/Mathias... reals, is there a counterexample to NP?
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A ‘consistency result’

Lemma (F., 2008)

If A is a counterexample to NP and forcing $\mathbb{P}$ adds a real then $\mathbb{P}$ forces that (the completion of) A has an irrep not equivalent to any ground-model irrep.

Corollary

‘There are no counterexamples to NP’ is relatively consistent with ZFC-Power set axiom.

Proof.

Add ORD many Cohen reals to $V$. 

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Add ORD many Cohen reals to $V$.  \qed
Once we have consistency of a positive answer to NP…”

Theorem (Glimm, 1960)

*If a simple separable C*-algebra A has inequivalent irreps, then it has $2^\aleph_0$ inequivalent irreps.*
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If a simple separable C*-algebra A has inequivalent irreps, then it has $2^\aleph_0$ inequivalent irreps.

Problem
What is the ‘right’ theorem for not necessarily separable algebras?
Once we have consistency of a positive answer to NP...

**Theorem (Glimm, 1960)**

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**Problem**

*What is the ‘right’ theorem for not necessarily separable algebras?*

**Question**

*For what cardinals (finite or infinite) $n$ there exists a simple C*-algebra with exactly $n$ inequivalent irreps?*

(Conjecture: $\Diamond \implies$ (at least) for all $n \in \mathbb{N}$.)
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Definition

A \( \varphi \in A^* \) is positive if \( \varphi(a) \geq 0 \) for positive \( a \in A \).

It is a state if \( \| \varphi \| = 1 \).

\[ S(A) := \{ \varphi \in A^* : \varphi \geq 0, \| \varphi \| = 1 \} \]

\( \mathcal{P}(A) := \) the extreme points of \( S(A) \) (pure states).
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**GNS-correspondence**

states $\iff$ cyclic representations.

pure states $\iff$ irreps.
A virgin problem

Theorem (Stone–Weierstrass)

If $A$ is a subalgebra of $B$, $B$ is abelian, and $A$ separates $\mathcal{P}(B) \cup \{0\}$ (i.e., $\varphi \neq \psi$ implies $\varphi \upharpoonright A \neq \psi \upharpoonright A$) then $A = B$. 

Problem (Noncommutative Stone–Weierstrass problem)

Is it true that if $A$ is a subalgebra of $B$ and $A$ separates $\mathcal{P}(B) \cup \{0\}$ then $A = B$?

A number of partial results (Glimm, Akemann, Sakai,...), all of them fairly old.

It is open even for separable C*-algebras.
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A number of partial results (Glimm, Akemann, Sakai, . . . ), all of them fairly old.

It is open even for separable C*-algebras.
3. $\mathbb{P}(C(H))$, the space of projections in $C(H)$

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\begin{align*}
p \leq q & \iff pq = p \\
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\begin{array}{ccc}
\mathcal{P}(\mathbb{N}) \ni X & \longrightarrow & p_X \in \mathcal{P}(\mathcal{B}(H)) \\
\downarrow & & \downarrow \\
\mathcal{P}(\mathbb{N})/\text{Fin} \ni [X] & \longrightarrow & [p_X] \in \mathcal{P}(\mathcal{C}(H))
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\begin{array}{ccc}
P(\mathbb{N}) \ni X & \longrightarrow & p_X \in \mathbb{P}(\mathcal{B}(H)) \quad p_X = \text{proj}_{\text{Span}\{e_n: n \in X\}} \\
\downarrow & & \downarrow \\
P(\mathbb{N})/\text{Fin} \ni [X] & \longrightarrow & [p_X] \in \mathbb{P}(\mathcal{C}(H)) \\
\subseteq^*\text{-chain} & \longrightarrow & \leq\text{-chain}
\end{array}
\]
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\downarrow & \quad \quad \downarrow \\
\mathcal{P}(\mathbb{N})/\text{Fin} \ni [X] & \quad \longrightarrow \quad [p_X] \in \mathbb{P}(C(H))
\end{align*}
\]

$\subseteq^\ast$-chain $\longrightarrow \leq$-chain

maximal $\subseteq^\ast$-chain $\not\rightarrow$ maximal $\leq$-chain

(E. Wofsey, 2007)
3. \( \mathcal{P}(\mathcal{C}(H)) \), the space of projections in \( \mathcal{C}(H) \)

\[
\begin{align*}
p \leq q & \iff pq = p \\
p \perp q & \iff pq = 0
\end{align*}
\]
\( \iff \|pq\| = 1 \)
\( \iff \|pq\| = 0 \)

Fix an orthonormal basis for \( H \), \( e_n \), for \( n \in \mathbb{N} \).

\[
\begin{array}{c}
\mathcal{P}(\mathbb{N}) \ni X \quad \xrightarrow{\text{}} \quad p_X \in \mathcal{P}(\mathcal{B}(H)) \\
\downarrow \quad \quad \quad \quad \downarrow \\
\mathcal{P}(\mathbb{N})/\text{Fin} \ni [X] \quad \xrightarrow{\text{}} \quad [p_X] \in \mathcal{P}(\mathcal{C}(H))
\end{array}
\]

\( \subseteq^* \)-chain \( \xrightarrow{\text{}} \) \( \leq \)-chain

maximal \( \subseteq^* \)-chain \( \not\rightarrow \) maximal \( \leq \)-chain \quad \text{(E. Wofsey, 2007)}

Similarly for (maximal) almost disjoint families. \text{(E. Wofsey, 2007.)}
Quantum cardinal invariants

Any statement $\mathcal{P}(\mathbb{N})/\text{Fin}$ has an analogue for $\mathcal{P}(C(H))$. ‘Quantized’ versions of cardinal invariants $a, b, c, d, \ldots$: $a^*, b^*, c^*, d^*, \ldots$
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**Problem**

*Gaps in $\mathcal{P}(\mathbb{N})/\text{Fin}$ vs. gaps in $\mathbb{P}(\mathcal{C}(H))$.*

(More on this in two weeks from now.)
3b. Quantum filters

Definition
Some $\mathcal{F} \subseteq \mathcal{P}(C(H)) \setminus \{0\}$ is a quantum filter if

$$(\forall p \in \mathcal{F})(\forall q)p \leq q \rightarrow q \in \mathcal{F}$$

$$(\forall F \subseteq \mathcal{F})F \text{ finite } \Rightarrow \| \prod F \| = 1$$
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Theorem (T. Bice, 2011)

The assertion ‘a maximal quantum filter in \( C(H) \) can be a filter’ is independent from ZFC.

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Does every ultrafilter \( U \) on \( \mathbb{N} \) generate a unique maximal quantum filter in \( C(H) \)?
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This is the famous Kadison–Singer Problem (1957).
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Does every pure state on the canonical copy of \( C(\beta\mathbb{N} \setminus \mathbb{N}) \) extend uniquely to a pure state of \( C(H) \)?
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Anderson, 1974: The conclusion of the KS problem is equivalent to an arithmetic statement.
Question

If $\mathcal{F}$ is a maximal quantum filter, is it diagonalized by the image of an ultrafilter $\mathcal{U}$?

More precisely: Is there a basis $e'_n$, for $n \in \mathbb{N}$, of $H$ such that $\{p^{(e'_n)}_X : X \in \mathcal{U}\} \subseteq \mathcal{F}$?
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Conjecture (Anderson, 1974)

*Every maximal quantum filter* \( \mathcal{F} \) *is generated by* \( \{ p^{(e'_n)}_X : X \in \mathcal{U} \} \) *for some* \( \mathcal{U} \) *and some* \( (e'_n) \).
Theorem (Akemann–Weaver, 2007)

*CH implies Anderson’s conjecture is false and KP#2 has a negative answer.*
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Conjecture (F.)

ZFC implies Anderson’s conjecture is false and KP\#2 has a negative answer.

Problem

Develop the analogue of Rudin–Keisler ordering for (maximal) quantum filters.

Problem

Prove analogues of Solecki’s results for ‘analytic quantum p-ideals.’

(Zamora–Aviles, 2009: Some results in this direction.)
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4. Multiplier algebras

A representation $\pi : A \to \mathcal{B}(H)$ is nondegenerate if

$$(\forall b \in \mathcal{B}(H)) b(\pi[A]) = \{0\} \text{ if and only if } b = 0.$$
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Fix a nondegenerate representation of \( A \), identify \( A \) with \( \pi(A) \).

\[
M(A) = \{ b \in \mathcal{B}(H) : bA \subseteq A \text{ and } Ab \subseteq A \}
\]

is a \( \text{C}^* \)-subalgebra of \( \mathcal{B}(H) \), called the \textit{multiplier algebra of} \( A \).
Properties of $M(A)$

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1. $M(A)$ does not depend on the choice of (nondegenerate) representation of $A$, up to the isomorphism.
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$C_0(A) = M(A) / A$ is the corona of $A$. 

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5. If $A = C_0(X)$ then $M(A) = C(\beta X)$ and $\mathcal{C}(A) = C(\beta X \setminus X)$

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Rigidity

An isomorphism $\Phi : C(A) \rightarrow C(B)$ is *trivial* if a *-homomorphism $F : M(A) \rightarrow M(B)$ lifts it:

$$
\begin{array}{ccc}
M(A) & \xrightarrow{\Phi} & M(B) \\
\downarrow & & \downarrow \\
C(A) & \xrightarrow{F} & C(B)
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$$

The following are relatively consistent with ZFC (and follow from PFA).

1. (Shelah) Every automorphism of $C(\beta \mathbb{N} \setminus \mathbb{N})$ is trivial.
2. (Veličkovic) Every automorphism of $C(\beta \kappa \setminus \kappa)$ is trivial, for all $\kappa$.
3. (F.) Every automorphism of $\bigotimes_{i<n} C(\beta \gamma \setminus \gamma)$ is trivial, for all $\gamma < \omega_1$ and all $n$.
4. (F.) Every automorphism of $C(H)$ is trivial.
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Quantum rigidity conjectures

Conjecture

*PFA implies that all *-isomorphisms between coronas of separable C*-algebras are trivial.*
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Theorem (F.–Hart, Coskey–F., 2011)

CH implies that $C(A)$ has $2^{\aleph_1}$ nontrivial automorphisms, for (almost) all separable $A$.
(More on this in two weeks from now.)
All presently known automorphisms $\Phi$ of $C(A)$’s are ‘pointwise trivial’: if $\Phi(a) = b$ then $a$ and $b$ are conjugate.
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**Question**

*Can $C(H)$ have an automorphism that is not pointwise trivial?*
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*Can $C(H)$ have an automorphism that is not pointwise trivial?*

The following is (provably) the most interesting instance of this question.

**Question (Brown–Douglas–Fillmore, 1977)**

*Is there an automorphism $\Phi$ of $C(H)$ that sends the unilateral shift to its adjoint?*
All presently known automorphisms Φ of C(A)’s are ‘pointwise trivial’: if Φ(a) = b then a and b are conjugate.

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The following is (provably) the most interesting instance of this question.

**Question (Brown–Douglas–Fillmore, 1977)**

*Is there an automorphism Φ of C(H) that sends the unilateral shift to its adjoint?*

**Lemma (F.)**

1. TA implies negative answer.
2. Such Φ cannot send the ‘atomic masa’ (i.e., the ‘canonical’ copy of $C(\beta\mathbb{N} \setminus \mathbb{N})$) to itself.
Topics I did not get to cover

1. Nonseparable C*-algebras (Weaver, F.–Katsura, F.)
5. Group cohomology and corona automorphisms (Coskey–F.)
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…and this is only the beginning.
More information available at
http://www.math.yorku.ca/~ifarah/preprints.html