Some ideals and ultrafilters on the natural numbers

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Ultrafilters on $\omega$

Topologically, ultrafilters on $\omega$ are points in $\beta\omega$.

Combinatorially, ultrafilters on $\omega$ are maximal downward directed upper sets in $\mathcal{P}(\omega)$.

$\mathcal{U} \subseteq \mathcal{P}(\omega)$ is an ultrafilter if

- $\mathcal{U} \neq \emptyset$ and $\emptyset \not\in \mathcal{U}$
- if $U_1, U_2 \in \mathcal{U}$ then $U_1 \cap U_2 \in \mathcal{U}$
- if $U \in \mathcal{U}$ and $U \subseteq V \subseteq \omega$ then $V \in \mathcal{U}$.
- for every $M \subseteq \omega$ either $M$ or $\omega \setminus M$ belongs to $\mathcal{U}$
Ultrafilters on $\omega$

Example. fixed (or principal) ultrafilter $\{A \subseteq \omega : n \in A\}$

Non-principal ultrafilters exist in ZFC.

Use of ultrafilters:

ultraproducts, parameters in some forcing constructions . . .
Some classes of ultrafilters

Definition.

For \( U, V \in \beta \omega \) we write \( U \leq_{RK} V \) if there exists \( f \in \omega^\omega \) such that \( \beta f(V) = U \).

\[
\beta f(V) = U \text{ iff } (\forall V \in V) f[V] \in U \text{ iff } (\forall U \in U) f^{-1}[U] \in V.
\]

- The relation \( \leq_{RK} \) is a quasiorder on \( \beta \omega \).
- Two ultrafilters \( U, V \) are equivalent, \( U \approx V \), if there exists a permutation \( \pi \) of \( \omega \) such that \( \beta \pi(V) = U \).
- The quotient relation defined by \( \leq_{RK} \) on \( \beta \omega / \approx \) is Rudin-Keisler order \( \leq_{RK} \).
Some classes of ultrafilters

Minimal elements in the Rudin-Keisler order on ultrafilters are selective ultrafilters.

Definition.

A free ultrafilter $\mathcal{U}$ is called a selective ultrafilter if for all partitions of $\omega$, $\{R_i : i \in \omega\}$, either for some $i$, $R_i \in \mathcal{U}$, or

$(\exists U \in \mathcal{U}) \ (\forall i \in \omega) \ |U \cap R_i| \leq 1.$

- $\mathcal{M}_\mathcal{U}$ adds a dominating real if $\mathcal{U}$ is a selective ultrafilter.
Some classes of ultrafilters

Definition.

A free ultrafilter \( \mathcal{U} \) is called a \( P \)-point if for all partitions of \( \omega \), \( \{ R_i : i \in \omega \} \), either for some \( i \), \( R_i \in \mathcal{U} \), or \( \exists U \in \mathcal{U} \) \( (\forall i \in \omega) |U \cap R_i| < \omega \).

- If \( \mathcal{V} \) is a \( P \)-point and \( \mathcal{U} \leq_{RK} \mathcal{V} \) then \( \mathcal{U} \) is a \( P \)-point.
- \( P \)-points were used for the first proof of the non-homogeneity of the space \( \omega^* \). (Rudin)
Some classes of ultrafilters

Definition.
An ultrafilter $\mathcal{U}$ on $\omega$ is called a nowhere dense ultrafilter if for every $f : \omega \to 2^\omega$ there exists $U \in \mathcal{U}$ such that $f[U]$ is nowhere dense.

- If $\mathcal{V}$ is a nowhere dense ultrafilter and $\mathcal{U} \leq_{RK} \mathcal{V}$ then $\mathcal{U}$ is nowhere dense.
- $\mathbb{I}_U$ adds no Cohen reals iff $\mathcal{U}$ is a nowhere dense ultrafilter. (Błaszczyk, Shelah)
Define a family of subsets $\mathcal{I}$ of a set $X$ such that $\mathcal{I}$ contains all singletons and is closed under subsets. An ultrafilter $\mathcal{U}$ on $\omega$ is called an $\mathcal{I}$-ultrafilter if for every $f : \omega \to X$ there exists $U \in \mathcal{U}$ such that $f[U] \in \mathcal{I}$.

Examples.

- nowhere dense ultrafilters . . . $X = 2^\omega$ and $\mathcal{I}$ are nowhere dense sets
- $P$-points . . . $X = 2^\omega$ and $\mathcal{I}$ are finite and converging sequences
$\mathcal{I}$-ultrafilters

Definition. (Baumgartner)

Let $\mathcal{I}$ be a family of subsets of a set $X$ such that $\mathcal{I}$ contains all singletons and is closed under subsets. An ultrafilter $\mathcal{U}$ on $\omega$ is called an $\mathcal{I}$-ultrafilter if for every $f : \omega \to X$ there exists $U \in \mathcal{U}$ such that $f[U] \in \mathcal{I}$.

Examples.

- nowhere dense ultrafilters \ldots $X = 2^\omega$ and $\mathcal{I}$ are nowhere dense sets
- $P$-points \ldots $X = 2^\omega$ and $\mathcal{I}$ are finite and converging sequences
  or $X = \omega \times \omega$ and $\mathcal{I} = \text{Fin} \times \text{Fin}$
- selective ultrafilters \ldots $X = \omega \times \omega$ and $\mathcal{I} = \mathcal{ED}$. 
**I-ultrafilters**

**Definition.** (Baumgartner)

Let $\mathcal{I}$ be a family of subsets of a set $X$ such that $\mathcal{I}$ contains all singletons and is closed under subsets. An ultrafilter $\mathcal{U}$ on $\omega$ is called an $\mathcal{I}$-ultrafilter if for every $f : \omega \to X$ there exists $U \in \mathcal{U}$ such that $f[U] \in \mathcal{I}$.

- wlog $\mathcal{I}$ is closed under finite unions i.e. $\mathcal{I}$ is an ideal
- $\mathcal{I}$-ultrafilters are downwards closed in $\leq_{RK}$
- if $\mathcal{I} \subseteq \mathcal{J}$ then every $\mathcal{I}$-ultrafilter is a $\mathcal{J}$-ultrafilter
- principal ultrafilters are $\mathcal{I}$-ultrafilters for every $\mathcal{I}$
Observation.

If $\mathcal{I}$ is a maximal ideal on $\omega$ then $\mathcal{U}$ is an $\mathcal{I}$-ultrafilter if and only if $\mathcal{I}^* \not\leq_R K \mathcal{U}$.

Theorem.

If $\mathcal{I}$ is a maximal ideal and $\chi(\mathcal{I}) = c$ then $\mathcal{I}$-ultrafilters exist in ZFC.

Proposition.

Assume $\mathcal{I}$ is ideal on $\omega$ and $\mathcal{U}$ is an ultrafilter on $\omega$. The following are equivalent:

- $\mathcal{U}$ is an $\mathcal{I}$-ultrafilter
- $\forall \mathcal{V} \not\leq_R \mathcal{U}$ for every ultrafilter $\mathcal{V} \supseteq \mathcal{I}^*$
All ideals contain the ideal Fin. All ultrafilters are non-principal.

Obviously, there are no Fin-ultrafilters.
**$\mathcal{I}$-ultrafilters for ideals on $\omega$**

All ideals contain the ideal Fin. **All ultrafilters are non-principal.**

Obviously, there are no Fin-ultrafilters.

An ideal $\mathcal{I}$ on $\omega$ is **tall** if for every $A \in [\omega]^\omega$ there exists an infinite set $B \subseteq A$ such that $B \in \mathcal{I}$.

**Proposition.**

If $\mathcal{I}$ is not a tall ideal then there are no $\mathcal{I}$-ultrafilters.
Theorem

(MA$_\sigma$-centered) If $\mathcal{I}$ is a tall ideal then $\mathcal{I}$-ultrafilters exist.
$\mathcal{I}$-ultrafilters for ideals on $\omega$

Theorem

$(\text{MA}_\sigma$-centered) If $\mathcal{I}$ is a tall ideal then $\mathcal{I}$-ultrafilters exist.

Theorem

If $\mathcal{I}$ is a (tall) $F_\sigma$-ideal or analytic $P$-ideal then every selective ultrafilter is an $\mathcal{I}$-ultrafilter.
Some ideals on $\omega$

Let $A$ be a subset of $\omega$ with an increasing enumeration $A = \{a_n : n \in \omega\}$. We say that $A$ is

**thin** if $\lim_{n \to \infty} \frac{a_n}{a_{n+1}} = 0$

$\mathcal{I}_{1/n} = \{A \subseteq \mathbb{N} : \sum_{a \in A} \frac{1}{a} < \infty\}$

$\mathcal{Z} = \{A \subseteq \mathbb{N} : \limsup_{n \to \infty} \frac{|A \cap n|}{n} = 0\}$
**Proposition.**
Every selective ultrafilter is a thin ultrafilter.

**Observation (Rudin).**
Every $P$-point is a $\mathcal{Z}$-ultrafilter.

**Theorem.**
$(MA_{\text{ctble}})$ For every $F_\sigma$-ideal $\mathcal{I}$ on $\omega$ there is a $P$-point that is not an $\mathcal{I}$-ultrafilter.
Theorem (Brendle).

\((\text{MA}_{\sigma\text{-centered}})\) There is a nowhere dense ultrafilter which is not \(\mathcal{Z}\)-ultrafilter.
\( \mathcal{I} \)-ultrafilters for ideals on \( \omega \)

**Theorem (Brendle).**

\( (\text{MA}_\sigma\text{-centered}) \) There is a nowhere dense ultrafilter which is not \( \mathcal{I} \)-ultrafilter.

**Theorem.**

\( (\text{MA}_{\text{ctble}}) \) There is a thin ultrafilter which is not nowhere dense.
Products of ultrafilters

Definition.

Let $\mathcal{U}$ and $\mathcal{V}_n$, $n \in \omega$, be ultrafilters on $\omega$. $\mathcal{U}$-sum of ultrafilters $\mathcal{V}_n$, $\sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega \rangle$, is an ultrafilter on $\omega \times \omega$ defined by $M \in \sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega \rangle$ if and only if $\{ n : \{ m : \langle n, m \rangle \in A \} \in \mathcal{V}_n \} \in \mathcal{U}$.

If $\mathcal{V}_n = \mathcal{V}$ for every $n \in \omega$ then we write $\sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega \rangle = \mathcal{U} \cdot \mathcal{V}$ and the ultrafilter $\mathcal{U} \cdot \mathcal{V}$ is called the product of ultrafilters $\mathcal{U}$ and $\mathcal{V}$. 
\(\mathcal{I}\)-sums

**Definition.** (Baumgartner)

Let \(\mathcal{C}\) and \(\mathcal{D}\) be two classes of ultrafilters. We say that \(\mathcal{C}\) is closed under \(\mathcal{D}\)-sums provided that whenever \(\{\mathcal{V}_n : n \in \omega\} \subseteq \mathcal{C}\) and \(\mathcal{U} \in \mathcal{D}\) then \(\sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega \rangle \in \mathcal{C}\).

- If \(\mathcal{D}\) is a class of \(\mathcal{I}\)-ultrafilters then we say that \(\mathcal{C}\) is closed under \(\mathcal{I}\)-sums.
$\mathcal{I}$-sums

No product (sum) of ultrafilters is a $P$-point.

Theorem (Baumgartner)
Nowhere dense ultrafilters are closed under nowhere dense sums.
Theorem.
Let $\mathcal{I}$ be a $P$-ideal on $\omega$ (or $\mathbb{N}$).
If $\mathcal{U}$ is an $\mathcal{I}$-ultrafilter and $\{n : \mathcal{V}_n \text{ is } \mathcal{I}\text{-ultrafilter}\} \in \mathcal{U}$ then
$$\sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega \rangle$$
is an $\mathcal{I}$-ultrafilter.

In other words, if $\mathcal{I}$ is a $P$-ideal then $\mathcal{I}$-ultrafilters are closed under $\mathcal{I}$-sums.
**Theorem.**

Let $\mathcal{I}$ be a $P$-ideal on $\omega$ (or $\mathbb{N}$). If $\mathcal{U}$ is an $\mathcal{I}$-ultrafilter and $\{ n : \mathcal{V}_n \text{ is } \mathcal{I}\text{-ultrafilter} \} \in \mathcal{U}$ then $\sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega \rangle$ is an $\mathcal{I}$-ultrafilter.

In other words, if $\mathcal{I}$ is a $P$-ideal then $\mathcal{I}$-ultrafilters are closed under $\mathcal{I}$-sums.

**Theorem.**

Let $\mathcal{I}$ be a $P$-ideal. If there is an $\mathcal{I}$-ultrafilter then there is an $\mathcal{I}$-ultrafilter that is not a $P$-point.
Proposition.
For arbitrary $\mathcal{U} \in \omega^*$ the ultrafilter $\mathcal{U} \cdot \mathcal{U}$ is not a thin ultrafilter.
Existence of $\mathcal{I}$-ultrafilters in ZFC

Theorem (Shelah)
It is consistent with ZFC that there are no nowhere dense ultrafilters.

Proposition.
It is consistent with ZFC that there are no thin ultrafilters.
Existence of $\mathcal{I}$-ultrafilters in ZFC

Theorem (Shelah)
It is consistent with ZFC that there are no nowhere dense ultrafilters.

Proposition.
It is consistent with ZFC that there are no thin ultrafilters.

Question.
Do $\mathcal{I}$-ultrafilters exist in ZFC for any tall analytic ideal $\mathcal{I}$? In particular, do $\mathcal{I}_{1/n}$-ultrafilters or $\mathcal{Z}$-ultrafilters exist in ZFC?
Weak \( \mathcal{I} \)-ultrafilters

Definition.

Let \( \mathcal{I} \) be an ideal on \( \omega \). An ultrafilter \( \mathcal{U} \) on \( \omega \) is called

- weak \( \mathcal{I} \)-ultrafilter if for each finite-to-one \( f : \omega \to \omega \) there exists \( U \in \mathcal{U} \) such that \( f[U] \in \mathcal{I} \).
- \( \mathcal{I} \)-friendly ultrafilter if for each one-to-one \( f : \omega \to \omega \) there exists \( U \in \mathcal{U} \) such that \( f[U] \in \mathcal{I} \).
Weak $\mathcal{I}$-ultrafilters

Definition.
Let $\mathcal{I}$ be an ideal on $\omega$. An ultrafilter $\mathcal{U}$ on $\omega$ is called

weak $\mathcal{I}$-ultrafilter if for each finite-to-one $f : \omega \to \omega$ there exists $U \in \mathcal{U}$ such that $f[U] \in \mathcal{I}$.

$\mathcal{I}$-friendly ultrafilter if for each one-to-one $f : \omega \to \omega$ there exists $U \in \mathcal{U}$ such that $f[U] \in \mathcal{I}$.

Theorem.
- (Gryzlov) $\mathcal{Z}$-friendly ultrafilters exist in ZFC.
- $\mathcal{I}_1/n$-friendly ultrafilters exist in ZFC.
$\mathcal{W}$-ultrafilters

The van der Waerden ideal $\mathcal{W}$ consists of subsets of $\omega$ which do not contain arbitrary long arithmetic progressions.
$\mathcal{W}$-ultrafilters

The van der Waerden ideal $\mathcal{W}$ consists of subsets of $\omega$ which do not contain arbitrary long arithmetic progressions.

- Every thin ultrafilter is a $\mathcal{W}$-ultrafilter.
- Every $\mathcal{W}$-ultrafilter is $\mathcal{Z}$-ultrafilter.
The van der Waerden ideal $\mathcal{W}$ consists of subsets of $\omega$ which do not contain arbitrary long arithmetic progressions.

- Every thin ultrafilter is a $\mathcal{W}$-ultrafilter.
- Every $\mathcal{W}$-ultrafilter is $\mathcal{Z}$-ultrafilter.

Questions.

- Do $\mathcal{W}$-ultrafilters exist in ZFC?
- Are $\mathcal{W}$-ultrafilters closed under products?